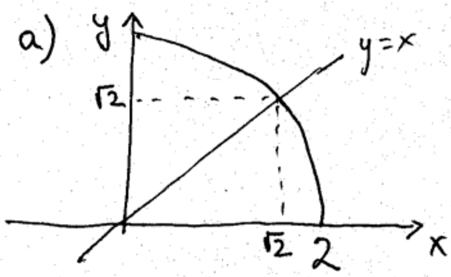
1. *(10 points)* Let R be the two dimensional region in the first quadrant, bounded by the lines $y=0$, $y=x$, and by an arc of the circle $x^2+y^2=4$.

a) Write down the x and y limits of integration for the double integral $\iint_R \cos(x^2+y^2)dxdy$;

b) Compute the integral in part a) by changing into polar coordinates.

a)



Two possibilities:

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \cos(x^2+y^2)\,dx\,dy$$

or

$$\int_0^{\sqrt{2}}\int_0^{x} \cos(x^2+y^2)\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} \cos(x^2+y^2)\,dy\,dx$$

b)

$$\int_0^{\pi/4}\int_0^{2} \cos(r^2)\,r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin(r^2)\Big|_0^2\,d\theta = \frac{\pi}{8}\sin 4.$$

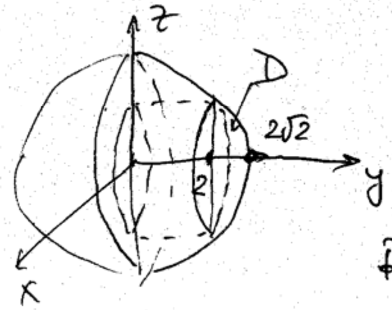
↳ use substitution $u=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

so $x^2+y^2=r^2$

2. *(10 points)* Find the volume of the solid spherical cap cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $y = 2$.



Shadow region onto the xz plane:

$y=2,\ x^2+2^2+z^2=8 \rightarrow x^2+z^2=4$

Rectangular coordinates:

$$\iiint_D 1\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2}^{\sqrt{4-x^2-z^2}} 1\ dy\,dz\,dx$$

The integral is easier to compute in cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ z = r\sin\theta \\ y = y \end{cases}$$

→ cylindrical coordinates with respect to y-axis (if you want to use the usual cylindrical coordinates, parametrize the spherical cap cut by the plane $z=2$).

Sphere in cylindrical coord: $r^2+y^2=8$.

$$\iiint_D 1\,dV = \int_0^{2\pi}\int_0^{2}\int_2^{\sqrt{8-r^2}} 1\ dy\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\left(\sqrt{8-r^2}-2\right)r\,dr\,d\theta$$

substitute $u=8-r^2$, $du=-2r\,dr$

(r,θ) parametrize the shadow region, $x^2+z^2\le 4$.

$$= \int_0^{2\pi}\left(\int_8^4(\sqrt{u}-2)\left(-\tfrac{1}{2}\right)du\right)d\theta$$

$$= 2\pi\cdot\frac{1}{2}\left(2u-\frac{2}{3}u^{3/2}\right)\Big|_8^4$$

$$= \frac{8\pi}{3}\left(4\sqrt{2}-5\right)$$

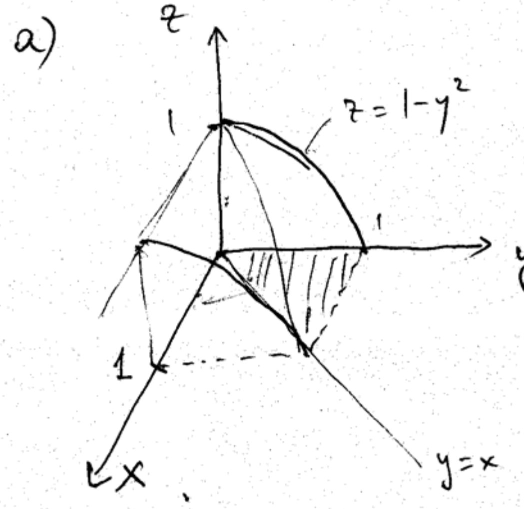
3. *(10 points)* Consider the triple integral:

$$\int_0^1 \int_x^1 \int_0^{1-y^2} \frac{e^z}{z-1} dz dy dx$$

a) Sketch the three dimensional region of integration and describe the surfaces that form its boundary.

b) Evaluate the integral by changing the order of integration to $dxdydz$

a)



First sketch the "shadow region" in the x-y plane $\left(\begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array} \right)$.
The 3D region lies above the shadow region, and under $z = 1-y^2$ (a cylinder).
So the 3D-region is bounded by the planes $z=0$ [redacted], $y = x$, $x=0$ and by cylinder $z = 1-y^2$

b) $\displaystyle\int_0^1 \int_0^{\sqrt{1-z}} \int_0^y \frac{e^z}{z-1} \, dx \, dy \, dz =$ ← First parametrize "shadow region" on the yz-plane

$$= \int_0^1 \int_0^{\sqrt{1-z}} \frac{y e^z}{z-1} dy \, dz = \int_0^1 \frac{e^z}{z-1} \cdot \frac{1-z}{2} dz = -\frac{1}{2} e^z \Big|_0^1 = -\frac{1}{2}(e-1)$$