Solution

Draw a picture to see that the rectangle of maximal area has 2 vertices on the $x = 1$ line and 2 on the circle. Let (x, y) be the coordinates of the vertex on the circle in the 1st quadrant.

We want to maximize the area, which is $2(x-1)y$ subject to the constraint $x^2 + y^2 = 3$. Lagrange multipliers give the equations

$$2y = 2\lambda x, \quad 2(x-1) = 2\lambda y, \quad x^2 + y^2 = 3.$$

$x \neq 0$ since $x \geq 1$, so $\lambda = y/x$. Substitute into second equation, we get $2x(x-1) = 2y^2$. From the 3rd, $y^2 = 3 - x^2$, thus $2x(x-1) = 6 - 2x^2$. Only good solution is $x = 3/2$ and then $y = \pm\sqrt{3}/2$.

The maximal area is then $\sqrt{3}/2$.

6. Let R be the volume given by $x^2 + y^2 + z^2 \leq 2z$, $x \geq 0$ and $z \geq 1$. Set up the integral

$$\int\int\int_R f dV$$

in rectangular, cylindrical and spherical coordinates. (You don't have to compute the integrals.)

Solution

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_1^{1+\sqrt{1-x^2-y^2}} f \ dzdydx.$$

$$\int_{-\pi/2}^{\pi/2} \int_0^1 \int_1^{1+\sqrt{1-r^2}} fr \ dzdrd\theta$$

$$\int_{-\pi/2}^{\pi/2} \int_0^{\pi/4} \int_{1/\cos\phi}^{2\cos\phi} f\rho^2 \sin\phi \ d\rho d\phi d\theta.$$

7. Compute the integral of z^n on the unit sphere $x^2 + y^2 + z^2 = 1$. (n is a positive integer.)

Solution

Use cylindrical coordinates. First we compute the upper hemisphere H and use its shadow which is the disc D given by $x^2 + y^2 \leq 1$.

$$\begin{array}{rcl} \int\int_H z^n \ d\sigma & = & \int\int_D z^n \frac{1}{|\cos\gamma|} r \ dr d\theta \\ & = & \int_0^{2\pi}\int_0^1 (\sqrt{1-r^2})^n \frac{1}{\sqrt{1-r^2}} r \ dr d\theta \\ & = & \int_0^{2\pi}\int_0^1 (1-r^2)^{(n-1)/2} r \ dr d\theta \\ & = & 2\pi \ (1-r^2)^{(n+1)/2} \frac{-2}{(n+1)2}\Big|_0^1 = \frac{2\pi}{n+1}. \end{array}$$

Now, if n is odd, on the lower hemisphere we get the same formula but with a minus sign, so the total integral is zero. If n is even, on the lower hemisphere we get the same formula with the same sign, so the total integral is $4\pi/(n+1)$.

8. Evaluate the double integral

$$\int\int_R \sqrt{25x^2 + 4y^2} dA$$

where R is the region enclosed by the ellipse $x^2/4 + y^2/25 = 1$. (Hint: Try to use substitutions.)

Solution

The substitution $x = 2u, y = 5v$ maps the unit disc $S := (u^2 + v^2 \leq 1)$ to the region R enclosed by the ellipse. Its Jacobian is 10. Thus

$$\begin{array}{rcl} \int\int_R \sqrt{25x^2 + 4y^2} dA & = & \int\int_S 10\sqrt{25(2u)^2 + 4(5y)^2} dA \\ & = & 100 \int\int_S \sqrt{u^2 + v^2} dA \\ & = & \int_0^{2\pi}\int_0^1 r \cdot r \ dr d\theta = \frac{200\pi}{3}. \end{array}$$

9. Find the flux of $\mathbf{F}(x, y, z) = (x^3 + \sin y, y^3 + e^z, z^3 - 1)$ across the hemisphere given by $x^2 + y^2 + z^2 = 4$ and $z \geq 0$. (For $\mathbf{n}$ choose the direction pointing away from the origin.)

Solution

Let V be the the upper half ball: all points inside the sphere $x^2+y^2+z^2 = 4$ and above the xy-plane, that is where $z \geq 0$. The boundary of V has 2 pieces, our hemisphere H and the disc D given by $x^2+y^2 \leq 4$ in the xy-plane. The divergence theorem says that

$$\int\int\int_V \nabla \cdot F \ dV = \int\int_H F \cdot \mathbf{n} d\sigma + \int\int_D F \cdot \mathbf{n} d\sigma.$$

Compute that $\nabla \cdot F = 3(x^2 + y^2 + z^2)$.