

**MAT 114, Spring 2007 - Third Midterm Solutions**

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1. Evaluate the integral

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} e^{x^2+y^2} dy dx.$$

A.  $\frac{3\pi}{8}(e-1)$   
D.  $\frac{3\pi}{4}(e^2-1)$

B.  $\frac{3\pi}{8}(e^4-1)$   
E.  $\frac{\pi}{8}(e^4-1)$

C.  $\frac{3\pi}{4}(e^2-1)$   
F.  $\frac{\pi}{4}(e-1)$

Solution. The region is half the quarter circle of radius 2 in the first quadrant. In polar coordinates it is given by:

$$\int_{\pi/4}^{\pi/2} \int_0^2 e^{r^2} r dr d\theta = \int_{\pi/4}^{\pi/2} \frac{e^4 - 1}{2} d\theta = \frac{\pi}{8}(e^4 - 1)$$

so the answer is E.

2. Find the area in the first quadrant enclosed by the curves  $y = x^2$ ,  $y = x^2/5$ ,  $xy = 2$ ,  $xy = 4$ , using the substitution  $u = x^2/y, v = xy$ .

A.  $\frac{1}{3} \ln 5$

B. 2

C. 8

D. 6

E. 4

F.  $\frac{2}{3} \ln 5$

Solution. The region in the  $uv$  plane is the rectangle enclosed by the lines  $u = 1$  and  $u = 5$  (that is  $x^2/y = 1$  and  $x^2/y = 5$ ), and  $v = 2$  and  $v = 4$  (that is  $xy = 2$  and  $xy = 4$ ).

To compute the Jacobian, we solve for  $x$  and  $y$  in terms of  $u$  and  $v$ :

$$x = u^{1/3}v^{1/3}, y = u^{-1/3}v^{2/3}.$$

The Jacobian is:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}u^{-2/3}v^{1/3} & \frac{1}{3}v^{-2/3}u^{1/3} \\ -\frac{1}{3}v^{2/3}u^{-4/3} & \frac{2}{3}v^{-1/3}u^{-1/3} \end{vmatrix} = \frac{2}{9}u^{-1} + \frac{1}{9}u^{-1} = \frac{1}{3u}.$$

So the area equals:

$$\int_1^5 \int_2^4 \frac{1}{3u} dv du = \frac{2}{3} \ln 5$$

and the answer is F.



4. Let  $D$  be the region bounded above by the sphere  $x^2 + y^2 + z^2 = 4$ , and below by the cone  $3z^2 = x^2 + y^2$ , for which  $z \geq 0$ .

a) (1pt) Sketch the region.

b) (3pts) Write down an integral giving the volume of  $D$  in cartesian coordinates.

c) (3pts) Write down an integral giving the volume of  $D$  in cylindrical coordinates.

d) (3pts) Write down an integral giving the volume of  $D$  in spherical coordinates.

You do not have to compute the integrals.

Solution. a) The region is an ice cream cone, inside the positive cone and under a spherical cap. The angle that the side of the cone makes with the  $z$ -axis is  $\pi/3$ .

b)

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{(x^2+y^2)/3}}^{\sqrt{4-x^2-y^2}} 1 dz dy dx$$

c)

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r/\sqrt{3}}^{\sqrt{4-r^2}} 1 dz r dr d\theta$$

d)

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^2 \sin \phi d\rho d\phi d\theta$$

5. Find the **surface area** of the spherical cap cut from the upper hemisphere  $x^2 + y^2 + z^2 = 2$  by the plane  $z = 1$ .

A.  $2\pi(\sqrt{2} - 1)$

B.  $4\pi$

C.  $2\pi(2 - \sqrt{2})$

D.  $4\pi(\sqrt{2} - 1)$

E.  $4\pi(2 - \sqrt{2})$

F.  $2\pi$

Solution. The region can also be described by the portion of the surface  $z = \sqrt{2 - x^2 - y^2}$  above the unit disk  $R$  in the  $xy$  plane. Therefore:

$$\begin{aligned} \text{Area} &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA = \\ &= \iint_R \sqrt{1 + \frac{x^2}{2 - x^2 - y^2} + \frac{y^2}{2 - x^2 - y^2}} dA \\ &= \iint_R \sqrt{\frac{2}{2 - x^2 - y^2}} dA \end{aligned}$$

We compute the last integral in polar coordinates:

$$\text{Area} = \int_0^{2\pi} \int_0^1 \sqrt{\frac{2}{2 - r^2}} r dr d\theta = \int_0^{2\pi} \sqrt{2}(-\sqrt{2 - r^2}) \Big|_0^1 d\theta = 2\pi\sqrt{2}(\sqrt{2} - 1) = 2\pi(2 - \sqrt{2})$$

so the answer is C.

6. Find the volume of the solid above the plane  $z = 0$ , below the cone  $z = \sqrt{x^2 + y^2}$  and inside the cylinder  $x^2 + y^2 = 1$ .

A.  $\frac{\pi}{3}$   
D.  $\frac{\pi}{2}$

B.  $\pi$   
E.  $2\pi$

C.  $\frac{2\pi}{3}$   
F.  $3\pi/2$

Solution. In cylindrical coordinates, the volume equals:

$$\int_0^{2\pi} \int_0^1 \int_0^r dzrdrd\theta = 2\pi/3$$

so the answer is C.

7. Consider the differential equation  $y' + 3y = 4e^{-x}$ . If  $y(x)$  is any solution of this equation, what is

$$\lim_{x \rightarrow \infty} e^x y(x)?$$

A. 0  
D. 3

B. 1  
E. 4

C. 2  
F.  $\infty$

Solution. This is a linear equation and the integrating factor is  $e^{3x}$ . Therefore:

$$e^{3x} y = \int 4e^{-x+3x} dx = 2e^{2x} + C, \text{ hence } y = 2e^{-x} + Ce^{-3x}.$$

The limit becomes:

$$\lim_{x \rightarrow \infty} e^x y(x) = \lim_{x \rightarrow \infty} 2 + Ce^{-2x} = 2$$

and the answer is C.

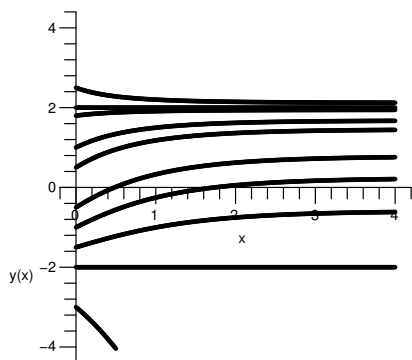
8. Match the following differential equations with the corresponding solution curves, by writing an equation number under each graph (you do not need to record your answer on the front page). Note: The solution curves do not intersect, even though it appears they do from the graphs.

1.  $\frac{dy}{dx} = \frac{y^2 - 4}{e^x}$

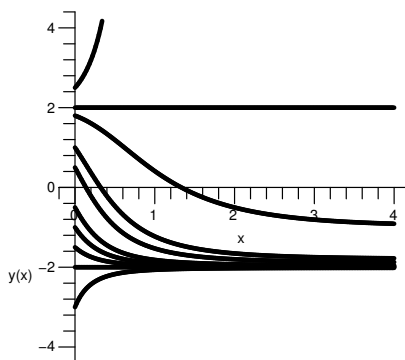
2.  $\frac{dy}{dx} = \frac{y^2 - 4}{3}$

3.  $\frac{dy}{dx} = \frac{4 - y^2}{3e^x}$

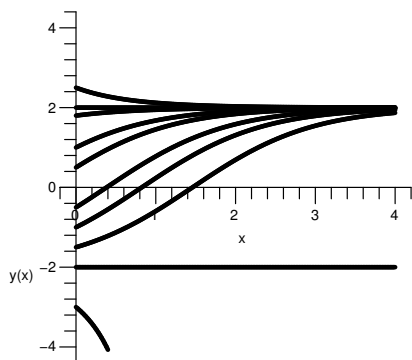
4.  $\frac{dy}{dx} = \frac{4 - y^2}{3}$



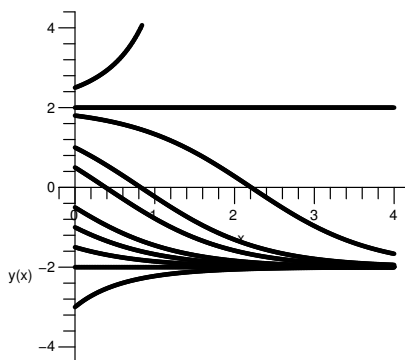
— 3 —



— 1 —



— 4 —



— 2 —

From the signs of  $dy/dx$ , it is clear that equations 1 and 2 correspond to the 2 graphs on the right. To find which one is which, notice that the slopes of the solution curves on the line  $y = 0$  change in the top right graph, but are the same in the bottom right graph. Therefore the top right graph corresponds to equation 1, where the slope depends on both  $y$  and  $x$ , while the bottom right graph corresponds to equation 2, where the slopes depend on  $y$  alone, thus being constant along horizontal lines.

The same reasoning applies to the other pair of graphs.

9. Consider a 10-ounce cup full of black coffee with no cream. At time  $t = 0$  we begin adding cream at the rate of 5 ounces/minute, while stirring. As cream is added, the cup overflows, so its total contents remains 10 ounces. How much cream is in the cup at time  $t = 2 \ln 2$ ?

A. 4 ounces

B. 5 ounces

C. 6 ounces

D. 7 ounces

E. 8 ounces

F. 9 ounces

Solution. Let  $f(t)$  be the amount of cream in the cup at time  $t$ . We have  $f(0) = 0$  and

$$\frac{df}{dt} = \text{Rate}_{\text{in}} - \text{Rate}_{\text{out}} = 5 - 5 \frac{f}{10}$$

(the rate at which cream is flowing in is given, while it flows out at 5 ounces/min times the concentration of cream in the cup,  $f/10$ )

Solving this equation (which is both a separable and a linear equation, so both methods can be used) gives:

$$f(t) = 10 + Ce^{-t/2}.$$

Since  $f(0) = 0$  we find  $C = -10$ , hence  $f(t) = 10(1 - e^{-t/2})$ . Therefore  $f(2 \ln 2) = 10(1 - 1/2) = 5$  ounces, and the answer is B.

10. Which of the following curves is perpendicular (normal) to all the curves in the family:

$$y = (x^2 + C)^{1/3}?$$

A.  $x^2 + y^2 = 1$

B.  $e^{2/y} = x^3$

C.  $xy = 1$

D.  $e^{-2y} = x^3$

E.  $y = e^x$

F.  $y = e^{-x}$

Solution.

$$\frac{dy}{dx} = \frac{1}{3}(x^2 + C)^{-2/3}2x = \frac{2x}{3y^2}$$

Therefore the normal curves have  $\frac{dy}{dx} = -\frac{3y^2}{2x}$ , which is a separable equation:

$$-\frac{dy}{y^2} = \frac{3dx}{2x}$$

$$\frac{1}{y} = \frac{3}{2} \ln|x| + C$$

or, after taking exponents:  $e^{2/y} = Ax^3$ . Therefore the answer is *B*. (for the constant  $A = 1$ ).