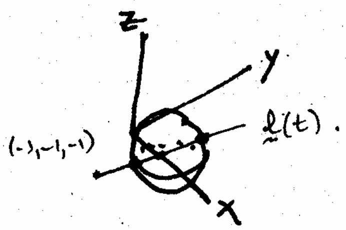
6. *(10 points)* Find the point(s) on the surface $x^2 - 2x + y^2 + z^2 = 0$ closest to the point $(-1, -1, -1)$.

METHOD I: Completion of the square gives the surface $\Rightarrow (x-1)^2 + y^2 + z^2 = 1$, the sphere of radius 1 and center $(1,0,0)$. Finding the point of minimal distance to $(-1,-1,-1)$ is equivalent to finding the points of intersection of the surface with the line $\underset{\sim}{\ell}(t) = t(-1,-1,-1) + (1-t)(1,0,0)$



we need $\text{dist}\left(\underset{\sim}{\ell}(t), (1,0,0)\right)^2 = 1$,

so

$$|\ell(t) - (1,0,0)|^2 = 1$$

$$= 4t^2 + t^2 + t^2 = 6t^2, \text{ hence}$$

$t = \pm \frac{1}{\sqrt{6}}$. To get the minimum, we want $t > 0$ as this puts us closer to $(-1,-1,-1)$

Hence

$$(x,y,z) = \frac{1}{\sqrt{6}}(-1,-1,-1) + \left(1 - \frac{1}{\sqrt{6}}\right)(1,0,0)$$

$$= \left(-\frac{2}{\sqrt{6}} + 1, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right).$$

METHOD II: minimize $(x+1)^2 + (y+1)^2 + (z+1)^2 = g$ subject to the constraint

$$f = x^2 - 2x + y^2 + z^2 = 0.$$

Lagrange multipliers:

$$\nabla g = \lambda \nabla f \Rightarrow \begin{aligned} 2(x+1) &= 2\lambda(x-1) \\ 2(y+1) &= 2\lambda y \\ 2(z+1) &= 2\lambda z \end{aligned}$$

$$\Rightarrow x = \frac{\lambda+1}{\lambda-1}, \quad y = z = \frac{1}{\lambda-1}$$

Substituting into $f = 0$:

$$\left(\frac{\lambda+1}{\lambda-1}\right)^2 - 2\left(\frac{\lambda+1}{\lambda-1}\right) + 2\left(\frac{1}{\lambda-1}\right)^2 = 0$$

$$(\lambda+1)^2 - 2(\lambda+1)(\lambda-1) + 2 = 0$$

$$-\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4+20}}{-2} = 1 \pm \sqrt{6}$$

so $x = \frac{2 \pm \sqrt{6}}{\pm\sqrt{6}} = 1 \pm \frac{2}{\sqrt{6}}$

$y = \frac{1}{\pm\sqrt{6}}, \quad z = \frac{1}{\pm\sqrt{6}}$

again for these to be closest to $(-1,-1,-1)$ they should be

$$x = 1 - \frac{2}{\sqrt{6}}, \quad y = \frac{-1}{\sqrt{6}}, \quad z = \frac{-1}{\sqrt{6}}.$$

5. *(12 points)* Consider the function

$$f(x,y) = x^3 - 3xy + y^3.$$

(a) Find all critical points of f. For each critical point, determine whether it is a local minimum, a local maximum or a saddle point, and write down the second degree Taylor polynomial (that is, the quadratic approximation) for f at that point.

(b) Find the absolute maximum and minimum of f on the square (containing its sides) in the xy-plane bounded by the lines: $x = 0, y = 0, x = 2, y = 2$.

(a) Critical pts: $\begin{cases} f_x = 3x^2 - 3y = 0 \\ f_y = 3y^2 - 3x = 0 \end{cases}$ $\quad \begin{rmatrix} y = x^2 \\ x = y^2 \end{rmatrix}$ $\Big\} \; x = (x^2)^2 = x^4$ $\Big\langle$ $x = 0,\ y = 0^2 = 0$; $x = 1,\ y = 1^2 = 1$ $\to$ $(0,0)$ & $(1,1)$

$f_{xx} = 6x \qquad f_{xy} = -3.$

$f_{yy} = 6y$

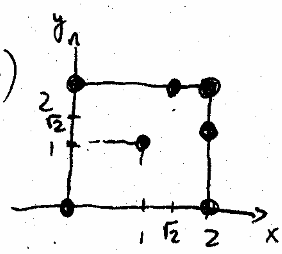
At $(0,0)$, $H(0,0) = -$[redacted] < 0 so $(0,0)$ is a saddle point.

At $(1,1)$, $H(1,1) = 6\cdot 6 - 3^2 = 27 > 0$, $f_{xx}(1,1) = 6 > 0$, so $(1,1)$ is a local min.

2nd Taylor polynomial at $(0,0)$: $f(x,y) \sim \frac{1}{2}(-6xy) = -3xy$

$(1,1)$: $f(x,y) \sim -1 + \frac{1}{2}\left[6(x-1)^2 - 6(x-1)(y-1) + 6(y-1)^2\right] = -1 + 3(x-1)^2 - 3(x-1)(y-1) + 3(y-1)^2$

(b)



Critical point inside: $(1,1) \rightsquigarrow f(1,1) = -1$

Critical points on the boundary:

- $x = 0$: $f(0,y) = y^3$, $0 \le y \le 2$, $\to (0,0)$ & $(0,2)$
- $y = 0$: by symmetry $\longrightarrow (0,0)$ & $(2,0)$.
- $x = 2$: $f(2,y) = y^3 - 6y + 8 \to f'(2,y) = 3y^2 - 6 = 0$, $y = \pm\sqrt{2}$

  $y = -\sqrt{2}$ not in the square, so the points are: $(2,0), (2,\sqrt{2}), (2,2)$
- $y = 2$: by symmetry, the critical pts. are $(0,2), (\sqrt{2},2), (2,2)$.

The maximum of f among the 7 critical points above occurs at $\boxed{f(0,2) = f(2,0) = 8}$, while the minimum occurs at $\boxed{f(1,1) = -1}$.

4. *(8 points)* The equation:

$$\ln(x+z) - ze^y + x^2 z = 0$$

determines z as an implicit function of the independent variables x and y. Compute $\frac{\partial z}{\partial y}$ at the point $(1,0,0)$.

METHOD I

$$\frac{1}{x+z}\frac{\partial z}{\partial y} - \left(ze^y + e^y\frac{\partial z}{\partial y}\right) + x^2\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}\left(\frac{1}{x+z} - e^y + x^2\right) = ze^y$$

$$\frac{\partial z}{\partial y} = \frac{ze^y}{\frac{1}{x+z} - e^y + x^2}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,0,0)} = \frac{0\cdot e^0}{\frac{1}{1+0} - e^0 + 1^2} = \frac{0}{1-1+1} = 0$$

METHOD II

$$F_z = \frac{1}{x+z} - e^y + x^2 \Rightarrow \left.F_z\right|_{(1,0,0)} = 1$$

$$F_y = -ze^y \Rightarrow \left.F_y\right|_{(1,0,0)} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = 0$$