MATH 201 - MIDTERM EXAM - SOLUTIONS

March 15 2006, 90 minutes

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1. *(15 points)* Find the following limits, if they exist, or show that they do not. Justify your answers.

   (a) $\displaystyle\lim_{(x,y)\to(1,-2)} \frac{x^2+y}{y^2+x}$

   Solution: $\displaystyle\lim_{(x,y)\to(1,-2)} \frac{x^2+y}{y^2+x} = \frac{(1+(-2)}{(-2)^2+1} = -\frac{1}{5}.$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4}$

   Solution:
   Along $y=0$: $\displaystyle\frac{x^4-y^4}{x^4+x^2y^2+y^4} = \frac{x^4}{x^4} = 1$
   Along $x=0$: $\displaystyle\frac{x^4-y^4}{x^4+x^2y^2+y^4} = -\frac{y^4}{y^4} = -1$
   So by the two path test, it has no limit at $(0,0)$.

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{2x^2+xy+2y^2}$

   Solution:
   Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4} &= \lim_{r\to0^+} \frac{r\cos\theta\sin^2\theta}{2r^2\cos^2\theta + r^2\cos\theta\sin\theta + 2r^2\sin^2\theta} \\ &= \lim_{r\to0^+} \frac{r\cos\theta\sin^2\theta}{2+\sin\theta\cos\theta}.\end{aligned}$$

   Then we notice that $|2+\sin\theta\cos\theta| \geq 1$ and $|\cos\theta\sin^2\theta| \leq 1$ therefore

$$|\frac{\cos\theta\sin^2\theta}{2+\sin\theta\cos\theta}| \leq 1$$

$$\Rightarrow |\frac{r\cos\theta\sin^2\theta}{2+\sin\theta\cos\theta}| \leq |r|$$

   and the limit is 0.

2. *(16 points)* In each of the following parts a pair of lines are given. If they intersect find the intersection point, otherwise find the distance between the lines. (By the distance between the lines we mean the length of the shortest path that connects them.)

   (a) $L_1(t) = (t, 2t, -t+1)$ and $L_2(t) = (t+1, 4t+6, -3t-4)$.

   Solution: We need to consider different parameters t, s for the lines and consider the following equations:

$$\begin{cases} t = s+1 \\ 2t = 4s+6 \\ -t+1 = -3s-4 \end{cases}$$

   Use the value of t from the first equation $t = s+1$ and in the second equation, we have

$$2s+2 = 4s+6 \quad \Rightarrow \quad s = -2 \quad \Rightarrow \quad t = -1.$$

   Then we can see that these values of t and s satisfy the third equation as well. Hence the lines DO intersect and the intersection point is $(-1, -2, 2)$.

   (b) $L_1(t) = (3t+3, -t, -t+2)$ and $L_2(t) = (t-1, 1-t, 2)$.

   Solution: Again look at the following system of equations:

$$\begin{cases} 3t+3 = s-1 \\ -t = 1-s \\ -t+2 = 2 \end{cases}$$

   By using the third and the second equations, one can see that $t = 0$ and $s = 1$. But these values don't satisfy the first equation and therefore the lines DO NOT intersect. Suppose $P = (3t+3, -t, -t+2)$ is a point of L_1 and $Q = (s-1, 1-s, 2)$ is a point of L_2. If PQ is the shortest path between these two lines then the vector

$$\overrightarrow{PQ} = \langle (s-1)-(3t+3), (1-s)-(-t), 2-(-t+2) \rangle = \langle s-3t-4, 1-s+t, t \rangle$$

   will be orthogonal to L_1 and L_2:

$$\begin{cases} \langle s-3t-4, 1-s+t, t \rangle \cdot \langle 3, -1, -1 \rangle = 0 \Rightarrow \quad 4s - 11t = 13 \\ \langle s-3t-4, 1-s+t, t \rangle \cdot \langle 1, -1, 0 \rangle = 0 \Rightarrow \quad 2s - 4t = 5 \end{cases}$$

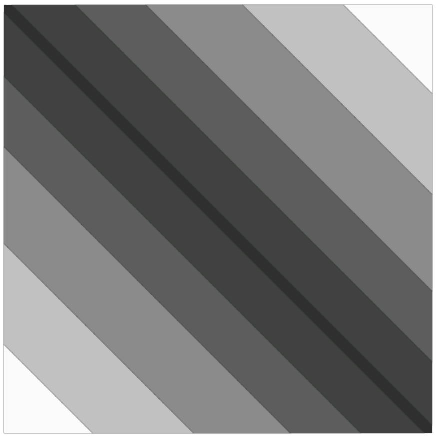
   We can solve the above equations for s and t and we have $t = -1$ and $s = \frac{1}{2}$ and the distance will be:

$$|\overrightarrow{PQ}| = |\langle -\frac{1}{2}, -\frac{1}{2}, -1 \rangle| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{\sqrt{6}}{2}.$$

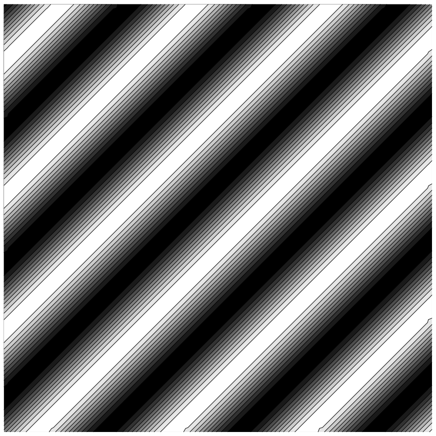
3. *(12 points)* Each figure below shows the set of level curves in the xy-plane for one of the functions

$$(i)\ f_1(x,y) = \sin(x - y^2) \qquad (ii)\ f_2(x,y) = \sin(x - y)$$
$$(iii)\ f_3(x,y) = x + y \qquad (iv)\ f_4(x,y) = x^2 + 2xy + y^2.$$

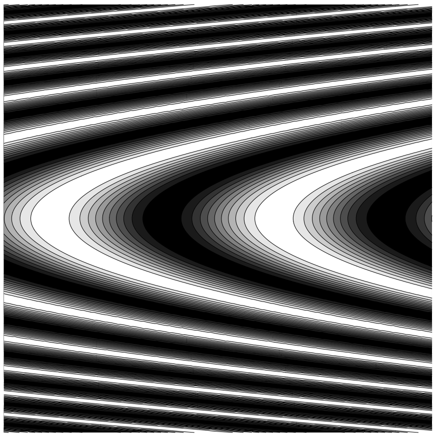
Note that darker regions correspond to the lower values of the functions in each picture and the origin is positioned in the center of the square. Indicate which picture represents each function. (You don't need to explain.)



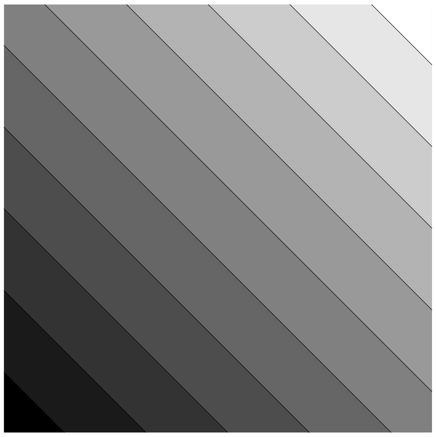
(a)



(b)



(c)



(d)

Solution: (i):(c) - (ii):(b) - (iii):(d) - (iv):(a).

4. *(18 points)* The temperature T at a point (x, y) in the plane is given implicitly as a function of x and y by the equation:

$$e^y T^2 + xye^T = 4.$$

(a) Compute $\partial T/\partial x$, $\partial T/\partial y$;

Solution: We take $\dfrac{\partial}{\partial x}$of both sides:

$$e^y(2T)\frac{\partial T}{\partial x} + ye^T + xye^T\frac{\partial T}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial T}{\partial x} = \frac{-ye^T}{2e^yT + xye^T}.$$

Samewise if we take $\dfrac{\partial}{\partial y}$ of both sides:

$$e^yT^2 + e^y(2T)\frac{\partial T}{\partial y} + xe^T + xye^T\frac{\partial T}{\partial y} = 0$$

$$\Rightarrow \frac{\partial T}{\partial y} = -\frac{e^yT^2 + xe^T}{2e^yT + xye^T}.$$

(b) An ant sits at the point $(0, 0)$, where the temperature is $T(0, 0) = 2$, and then she starts moving in the direction of the vector $\langle 3, 4\rangle$. At the moment it leaves the point $(0, 0)$, does the ant feel an increase or a decrease in temperature?

Solution: $\bigtriangledown T(0,0) = \langle \frac{\partial T}{\partial x}(0,0), \frac{\partial T}{\partial y}(0,0)\rangle = \langle 0, -\frac{2^2+0}{2\cdot 2+0}\rangle = \langle 0, -1\rangle$. Let

$$\mathbf{u} = \frac{\langle 3,4\rangle}{|\langle 3,4\rangle|} = \langle \frac{3}{5}, \frac{4}{5}\rangle$$

$$D_{\mathbf{u}}T(0,0) = \bigtriangledown T(0,0)\cdot \mathbf{u} = \langle 0, -1\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = -\frac{4}{5} < 0.$$

So the directional derivative of T at $(0, 0)$ in the direction of $\langle 3, 4\rangle$ is negative. Therefore the ant feels a decrease in the temperature.

5. *(20 points)* Let $f(x,y) = x^3 + y^3 + 3xy + 1$.

(a) Find all critical points of f and classify them as local minima, local maxima or saddle points.

Solution:

$$\begin{cases} f_x = 3x^2 + 3y = 0 & \Rightarrow \qquad y = -x^2 \\ f_y = 3y^2 + 3x = 0 & \Rightarrow \qquad x = -y^2 \end{cases}$$

Hence $x = -x^4$ and either $x = 0$ and $y = 0$ or $x = -1$ and $y = -1$. So the critical points are $(0,0)$ and $(-1,-1)$. On the other hand

$$f_{xx} = 6x, \quad f_{xy} = 3, \quad f_{yy} = 6y$$

$$H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 3 \\ 3 & 6y \end{vmatrix}$$

So $H(0,0) = -9 < 0$ and $(0,0)$ is a saddle point and $H(-1,-1) = 27 > 0$ and because $f_{xx}(-1,-1) < 0$, the point $(-1,-1)$ is a local maximum.

(b) Find the absolute extrema of f on the triangular region enclosed by the lines $x = -2$, $y = -2$ and $x + y = 0$.

Solution: At the critical points we have $f(0,0) = 1$ and $f(-1,-1) = 2$. We need to look at the boundary as well. The boundary has three segments:

Side A: the segment that connects $(-2,-2)$ to $(-2,2)$. On this side $x = -2$ and $-2 \leq y \leq 2$. So on this side $f = f(-2,y) = -8 + y^3 - 6y + 1 = y^3 - 6y - 7$. To find its critical points, we have

$$3y^2 - 6 = 0 \quad \Rightarrow \quad y = \pm\sqrt{2}.$$

This gives two points $(-2,\sqrt{2})$ and $(-2,-\sqrt{2})$. The value of f at these points is:

$$f(-2,\sqrt{2}) = -4\sqrt{2} - 7, \qquad f(-2,-\sqrt{2}) = 4\sqrt{2} - 7.$$

Side B: the segment that connects $(-2,-2)$ to $(2,-2)$. On this side $y = -2$ and $-2 \leq x \leq 2$. So on this side $f = f(x,-2) = x^3 - 6x - 7$. Similar to above, we can see that the critical points on this side are $(\sqrt{2},-2)$ and $(-\sqrt{2},-2)$ and the value of f at these points is:

$$f(\sqrt{2},-2) = -4\sqrt{2} - 7, \qquad f(-\sqrt{2},-2) = 4\sqrt{2} - 7.$$

Side C: the segment that connects $(-2,2)$ to $(2,-2)$. On this side $y = -x$ and $-2 \leq x \leq 2$. So on this side $f = f(x,-x) = x^3 - x^3 - 3x^2 + 1 = 1 - 3x^2$. To find the critical points on this side we have:

$$-6x = 0 \quad \Rightarrow \quad x = y = 0,$$

which gives only one critical point $(0,0)$ which we have already considered.
Finally we need to look at the value of f on the vertices of the triangle:

$$f(-2,2) = f(2,-2) = -11, \quad \text{and} \quad f(-2,-2) = -3.$$

Comparing the obtained values above, we see that the absolute maximum of f happens at

$$f(-1,-1) = 2$$

and the absolute minimum of f happens at

$$f(-2,\sqrt{2}) = f(\sqrt{2},-2) = -4\sqrt{2} - 7.$$

6. *(19 points)* Find the absolute minimum of the function $f(x, y) = (x + y + 2)^2$ on $x^2 + y^2 \leq 1$.

Solution: First we should look for critical points:

$$\begin{cases} f_x = 2(x + y + 2) \\ f_y = 2(x + y + 2) \end{cases}$$

This implies that $x + y = -2$. But the line $x + y = -2$ does not intersect the set $x^2 + y^2 \leq 1$ and therefore we don't need to consider any of these critical points.

To find the minimum and the maximum on the boundary, we use the Lagrange Multipliers Method for f and with the constraint $g(x, y) = x^2 + y^2 - 1 = 0$:

$$\begin{cases} \bigtriangledown f = \lambda \bigtriangledown g \\ g = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \langle 2(x + y + 2), 2(x + y + 2) \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 1 \end{cases}$$

$$\Rightarrow \quad x + y + 2 = \lambda x = \lambda y$$

Hence either $\lambda = 0$ or $x = y$. If $\lambda = 0$ then $x + y + 2 = 0$ but as we explained before there is no such point in the set $x^2 + y^2 \leq 1$. Hence $x = y$; but

$$x^2 + y^2 = 1 \quad \Rightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}.$$

This gives two points $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The value of f at these points is:

$$f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = (\sqrt{2} - 2)^2 \quad \text{and} \quad f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\sqrt{2} + 2)^2.$$

Therefore the absolute minimum of f is $(\sqrt{2} - 2)^2$ and its absolute maximum is $(\sqrt{2} + 2)^2$.