Lecture 2 - Lie Groups, Lie Algebras, and Geometry

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1 Overview

If D is any linear operator on a vector space, we can define $Exp(D)$ by

$$Exp(D) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} D^n. \tag{1}$$

The sum converges if the operator is bounded. In other cases, such as differential operators on Sobolev spaces, one has to deal with convergence on a case-by-case basis or work with densely defined operators. If A and B are commuting operators, we have

$$Exp(A)\,Exp(B) \;=\; Exp(A+B). \tag{2}$$

But the situation is bad in the non-commutative case. Expanding in terms of Taylor series shows that

$$Exp(A)\,Exp(B) \;=\; Exp(A+B) \;+\; O(2) \tag{3}$$

where $O(2)$ consists of terms of quadratic and higher orders in A and B. It is a simple matter to compute that in fact

$$Exp(A)\,Exp(B) \;=\; Exp(A+B+\frac{1}{2}[A,B] \;+\; O(3)) \tag{4}$$

The Campbell-Baker-Hausdorff theorem asserts that $O(3)$ can be written in terms of only bracket terms (things like $[A,[A,B]]$, $[[B,[A,B]],B]$ and so forth) without terms like A^2 or ABA.

2 Linear Lie Groups and Differential Geometry

A *linear Lie group* G is a Lie group whose underlying topological space is a set of differentiably varying matrices, with the group action being matrix multiplication. If $Id \in G$ is the identity, derivatives of paths through the identity can be identified with matrices, and the Lie derivative is simply the commutator of these matrices. In this way $\mathfrak{g} = T_{Id}G$ is a Lie algebra of matrices.

The principle examples are the

$$GL(n,\,\mathbb{F}) \;=\; \left\{ \, M_{n\times n} \,\middle|\, det(M_{n\times n}) \neq 0 \, \right\} \tag{5}$$

where $\mathbb{F}$ is the base field $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (there is no analogue for the octonions). Via realification, we need only really consider the case of base field $\mathbb{R}$, although working with base fields $\mathbb{R}$ or $\mathbb{H}$ is conceptually and computationally simpler in many cases. The groups $Gl(n,\mathbb{R})$ are of course non-compact; however they have a large number of important subgroups:

$$SL(n,\,\mathbb{R}) \;=\; \left\{ \, M \in GL(n,\mathbb{R}) \,\middle|\, det(M) \;=\; 1 \, \right\} \tag{6}$$

which is still non-compact, but which has the compact subgroup

$$SO(n,\mathbb{R}) \;=\; \left\{ \, M \in GL(n,\mathbb{R}) \,\middle|\, M^T M \;=\; Id \, \right\}. \tag{7}$$

A second compact subgroup, in the even dimensional case, is the symplectic group

$$SP(2n,\,\mathbb{R}) \;=\; \left\{ M \in GL(n,\mathbb{R}) \,\middle|\, M^T J M \;=\; J \right\} \tag{8}$$

where

$$J \;=\; \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \tag{9}$$

The Sp groups are related to both symplectic 2-forms and to unitary transformations of quaternionic vector spaces. Under the commutator bracket, we have the corresponding real Lie algebras

$$\begin{aligned}
\mathfrak{gl}(n,\mathbb{R}) \;&=\; \text{all real } n\times n \text{ matrices} \\
\mathfrak{sl}(n,\mathbb{R}) \;&=\; \left\{ \, x \in \mathfrak{gl}(n,\mathbb{R}) \,\middle|\, tr(x) = 0 \, \right\} \\
\mathfrak{o}(n,\mathbb{R}) \;&=\; \left\{ \, x \in \mathfrak{gl}(n,\mathbb{R}) \,\middle|\, x^T \,+\, x = 0 \, \right\} \\
\mathfrak{sp}(2n,\mathbb{R}) \;&=\; \left\{ \, x \in \mathfrak{gl}(2n,\mathbb{R}) \,\middle|\, x^T J \,+\, Jx = 0 \, \right\}
\end{aligned} \tag{10}$$

The "realification" process alluded to above is the process of taking a complex $n\times n$ matrix $C = A + iB$ and writing it as real $2n \times 2n$ matrix

$$C \;=\; \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{11}$$

and noting that the algebra laws carry over. However, if one wishes to work directly with complex matrices (as one often does), we have the groups

$$\begin{aligned}
GL(n,\,\mathbb{C}) \;&=\; \text{Invertible } n\times n \text{ complex matrices} \\
SL(n,\,\mathbb{C}) \;&=\; \left\{ \, M \in GL(n,\mathbb{C}) \,\middle|\, det(M) \;=\; 1 \, \right\} \\
SO(n,\mathbb{C}) \;&=\; \left\{ \, M \in GL(n,\mathbb{C}) \,\middle|\, M^T M \;=\; Id \right\} \\
SP(2n,\,\mathbb{C}) \;&=\; \left\{ M \in GL(n,\mathbb{C}) \,\middle|\, M^T J M \;=\; Id \right\}
\end{aligned} \tag{12}$$

where J is the same as before. In addition we have the very important unitary and special unitary groups

$$\begin{aligned} U(n) &= \left\{ M \in GL(n, \mathbb{C}) \,\middle|\, \overline{M}^T M = 1 \right\} \\ SU(n) &= U(n) \cap SL(n, \mathbb{C}). \end{aligned} \tag{13}$$

We easily see that

$$\begin{aligned} \mathfrak{gl}(n, \mathbb{C}) &= \mathfrak{gl}(n, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\ \mathfrak{sl}(n, \mathbb{C}) &= \mathfrak{sl}(n, \mathbb{R}) \otimes \mathbb{C} \\ \mathfrak{o}(n, \mathbb{C}) &= \mathfrak{o}(n, \mathbb{R}) \otimes \mathbb{C} \\ \mathfrak{sp}(n, \mathbb{C}) &= \mathfrak{sp}(n, \mathbb{R}) \otimes \mathbb{C}. \end{aligned} \tag{14}$$

However, the unitary and special unitary algebras are real algebras

$$\begin{aligned} \mathfrak{u}(n) &= \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \,\middle|\, \overline{x}^T + x = 0 \right\} \\ \mathfrak{su}(n) &= \left\{ x \in \mathfrak{sl}(n, \mathbb{C}) \,\middle|\, \overline{x}^T + x = 0 \right\}, \end{aligned} \tag{15}$$

as the condition $\overline{x}^T + x = 0$ is not preserved by complex multiplication. However

$$\begin{aligned} \mathfrak{gl}(\mathfrak{n}, \mathbb{C}) &= \mathfrak{u}(n) \otimes \mathbb{C} \\ \mathfrak{sl}(\mathfrak{n}, \mathbb{C}) &= \mathfrak{su}(n) \otimes \mathbb{C} \end{aligned} \tag{16}$$

Many other special relations carry over; for instance unitary matrices are precisely the matrices that are both symplectic and orthogonal. If we define

$$Sp(n, \mathbb{H}) = \left\{ W \in GL(n, \mathbb{H}) \,\middle|\, \overline{W}^T W = 1 \right\} \tag{17}$$

then $Sp(n, \mathbb{H})$ is naturally embedded in $SP(2n, \mathbb{C})$ as a compact subgroup, and $Sp(2n, \mathbb{C})/Sp(n, \mathbb{H})$ is contractible (so $Sp(n, \mathbb{H})$ is a maximal compact subgroup).

2.1 Exponential maps

A linear Lie group G has two exponential maps from $\mathfrak{g}$ to G, the first, denoted "exp" defined using the flow of the identity under left-invariant vector fields, and the other, denoted "Exp" defined using the exponential operator. These are easily seen to be the same operator. Note that for $v \in \mathfrak{g}$, we have $\frac{d}{dt}\big|_{t=0} Exp(tv) = v$, so at time 0 the paths $t \mapsto Exp(tv)$ and $t \mapsto exp(tv)$ share the same initial vector. Then note that the tangent vectors to both paths are invariant under push-forward by left-translation.

2.2 Riemannian Geometry

Let G be a Lie group with Riemannian metric g. Obviously g can be made left-invariant by placing a non-degenerate bilinear form g on $\mathfrak{g}$ and then requiring $g_p(v, w) = g(L_{p^{-1}*}v, L_{p^{-1}*}w)$.

Therefore, with g invariant under pullback along left-translations, the right-invariant fields are Killing fields.

Any compact Lie group has a bi-invariant metric. Such a metric is characterized by being both left-invariant, and having left-invariant fields as Killing fields. Thus

$$g([v,w],z) \ = \ g(v,[w,z]) \tag{18}$$

for left-invariant fields v, w, and z (aka "associativity"). In this case, the Koszul formula

$$\begin{aligned} 2g\left(\nabla_x y,\, z\right) \ &= \ ([x,y],\, z) \,-\, ([y,z],\, x) \,+\, ([z,x],\, y) && (19) \\ &= \ ([x,y],\, z) && (20) \end{aligned}$$

shows that

$$\nabla_x y = \frac{1}{2}[x,y]. \tag{21}$$

In particular the left-invariant fields integrate out to geodesics. Thus the exponential map from Lie group theory is the same as the exponential map of Riemannian geometry.

3 Examples

3.1 $SU(2)$

For certain reasons, this may be the most important example of a compact Lie group. Matrices $M \in \mathbb{C}(2)$ are unitary if $\overline{M}^T M = Id$ and special if $det(M) = 1$. These are the matrices of the form

$$M \ = \ \left(\begin{array}{cc} z & -\overline{w} \\ w & \overline{z} \end{array}\right) \qquad \text{where} \qquad |z|^2 + |w|^2 \ = \ 1 \tag{22}$$

The standard topology gives this group the differentiable structure of $\mathbb{S}^3$. The Lie algebra $\mathfrak{su}(2)$ is the real span of the three trace-free antihermitian matrices

$$\begin{aligned} \vec{x}_1 \ &= \ \sqrt{-1}\sigma_1 \ = \ \sqrt{-1}\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) && (23) \\ \vec{x}_2 \ &= \ \sqrt{-1}\sigma_2 \ = \ \sqrt{-1}\left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right) && (24) \\ \vec{x}_3 \ &= \ \sqrt{-1}\sigma_3 \ = \ \sqrt{-1}\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) && (25) \end{aligned}$$

which is the Lie algebra of purely imaginary quaternions under the commutator bracket, which is isomorphic to the cross product algebra.

Computing the adjoint maps on the Lie algebra $\mathfrak{su}(2)$, we find

$$\begin{aligned} ad_{\vec{x}_1} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \\ ad_{\vec{x}_2} &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \\ ad_{\vec{x}_3} &= \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{26}$$

so the Killing form is the negative definite bilinear form

$$\kappa = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix} \tag{27}$$

and we put the bi-invariant Riemannian metric $g = -\frac{1}{8}\kappa$ on $SU(2)$, which gives the group constant curvature $+1$.

3.1.1 Hopf Fibration

The vectors $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ are represented by paths through the identity $\tau \mapsto Exp(\tau x_i)$. Moving by left translation, at a point $M \in SU(2)$ the representative paths are $\tau \mapsto MExp(\tau x_i)$, which are the matrices Mx_i. Moving by right translation, at a point $M \in SU(2)$ the representative paths are $\tau \mapsto MExp(\tau x_i)$, which are the matrices $x_i M$.

To get a feel for the difference between left- and right-invariant vector fields, consider the association between $SU(2)$ and the set $\mathbb{S}^3 \in \mathbb{C}^2$:

$$\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \quad \longleftrightarrow \quad (z,\, w) \;\in\; \mathbb{C}^2. \tag{28}$$

The left-invariant vector field from $\vec{x}_3$ gives rise to the diffeomorphisms by right-translation $\varphi_\tau(M) = R_{Exp(\tau\sqrt{-1}\sigma_3)}M = MExp(-\tau\sqrt{-1}\sigma_3)$, or

$$\varphi_\tau \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} = \begin{pmatrix} ze^{i\tau} & -\overline{we^{i\tau}} \\ we^{i\tau} & \overline{ze^{i\tau}} \end{pmatrix} \qquad or \qquad \varphi_\tau(z,w) = (ze^{i\tau},\, we^{i\tau}) \tag{29}$$

so that flow lines of the left-invariant field $\vec{x}_3$ correspond to fibers of the Hopf map $(z,w) \mapsto zw^{-1}$.

The right-invariant vector field from $\vec{x}_3$ gives rise to the diffeomorphisms by left-translation $\varphi_\tau(M) = L_{Exp(-\tau\sqrt{-1}\sigma_3)}M = Exp(-\tau\sqrt{-1}\sigma_3)M$, or

$$\varphi_\tau \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} = \begin{pmatrix} ze^{i\tau} & -\overline{we^{-i\tau}} \\ we^{-i\tau} & \overline{ze^{i\tau}} \end{pmatrix} \qquad or \qquad \varphi_\tau(z,w) = (ze^{i\tau},\, we^{-i\tau}) \tag{30}$$

so that flow lines of the left-invariant field $\vec{x}_3$ correspond to fibers of the anti-Hopf map $(z, w) \mapsto z\overline{w}^{-1}$.

3.1.2 $SU(2)$ as a spin group

Now consider an arbitrary trace-free anti-hermtian matrix, which can be interpretted as a vector in $\mathfrak{su}(2)$:

$$\vec{v} \;=\; a\vec{x}_1 + b\vec{x}_2 + c\vec{x}_3 \;=\; \begin{pmatrix} -ci & -b-ci \\ b-ci & ci \end{pmatrix}. \tag{31}$$

Note that $det(\vec{v}) = a^2 + b^2 + c^2 = |v|^2 = -\frac{1}{8}\kappa(v, v)$ is the norm-square of the vector. The metric is then

$$g(\vec{v}, \vec{w}) \;=\; \frac{1}{2}\left(det(\vec{v} + \vec{w}) \;-\; det(\vec{v}) \;-\; det(\vec{w})\right). \tag{32}$$

In particular, if $M \in SU(2)$ (so in particular conjugation preserves), we have that $Ad_M : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is in fact orthogonal. Therefore

$$Ad : SU(2) \;\to\; SO(3). \tag{33}$$

Because $SU(2)$ is connected, the image is in a connected subgroup $O(3)$, so we have a Lie algebra epimorphism The kernel of the Ad map is easily seen to be $\pm Id$, giving a 2-1 covering map; indeed this is a universal covering map of $SO(3)$, as $SU(2)$ is simply-connected.

The double cover of a special orthogonal group $SO(n)$ is called its associated *spinor group*, denoted $Spin(n)$. We therefore have $Spin(3) = SU(2)$.

3.2 $SL(2, \mathbb{R})$

For certain reasons, $SL(2, \mathbb{R})$ may be the most important example of a non-compact Lie group. Considering its Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = span_{\mathbb{R}}\{h, x, y\}$, we have

$$\begin{aligned} ad_h \;&=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\ ad_x \;&=\; \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ ad_y \;&=\; \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \end{aligned} \tag{34}$$

so that

$$\kappa \;=\; \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \tag{35}$$

In analogy with the $SU(2)$ case, a good bi-invariant metric is

$$g \;=\; -\frac{1}{8}\kappa \tag{36}$$

which is Lorenzian, of signature $(+,-,-)$ (a time-like unit vector is $x-y$, two space-like unit vectors are h and $x+y$). Setting $\vec{v} = a\vec{h} + b\vec{x} + c\vec{y}$ again we compute

$$\det(\vec{v}) \;=\; -a^2 - bc \;=\; -\frac{1}{8}\kappa(\vec{v},\,\vec{v}) \;=\; \|\vec{v}\|^2 \tag{37}$$

so that

$$g(\vec{v},\,\vec{w}) \;=\; \frac{1}{2}\left(\det(\vec{v}+\vec{w}) \;-\; \det(\vec{v}) \;-\; \det(\vec{w})\right). \tag{38}$$

Because any other bi-invariant metric gives rise to an associative bilinear form on $\mathfrak{sl}_2$, and because any two associative forms are equal up to constant multiplication (which must be real if it comes from a metric), it follows that $SL(2,\mathbb{R})$ has no bi-invariant Riemannian metric.

Now consider the adjoint representation $Ad : SL(2,\mathbb{R}) \to Hom(\mathfrak{sl}(2,\mathbb{R}))$. We have that conjugation with $M \in SL(2,\mathbb{R})$ preserves determinant, so therefore preserves the metric, giving us a map into $O(1,3)$, which is easily seen to be a surjection

$$Ad \,:\, SL(2,\mathbb{R}) \;\to\; SO^+(1,2) \tag{39}$$

into the orthochronous special orthogonal group. The kernel is, again, seen to be $\pm Id$, so this is a 2-1 covering map. Therefore

$$SL(2,\mathbb{R}) \;\approx\; Spin(1,2). \tag{40}$$

Note however $SL(2,\mathbb{R})$ is not simply-connected. Its Lorenzian metric is Einsteinian with scalar curvature -2, so its universal cover is a model of $(1+2)$-dimensional anti-deSitter space (an empty universe with negative cosmological constant, or the Lorenzian analogue of hyperbolic space).

Of course $SL(2,\mathbb{R})$ has left-invariant Riemannian metrics; the natural geometry is of an $\mathbb{S}^1$-bundle over hyperbolic space, and is one of Thurston's eight model geometries.

3.3 $SL(2,\mathbb{C})$

We can prove the existence of a 2-1 map $SL(2,\mathbb{C}) \to SO^+(1,3)$ (also implying that $SO^+(1,3) \approx PSL(2,\mathbb{C})$). Start with any anti-Hermitian matrix

$$\vec{x} \;=\; \begin{pmatrix} -i(t+z) & -y-ix \\ y-ix & -i(t-z) \end{pmatrix} \;=\; t\,\vec{x}_0 \;+\; x\,\vec{x}_1 \;+\; y\,\vec{x}_2 \;+\; z\,\vec{x}_3 \tag{41}$$

(under the commutator bracket, this is the Lie algebra of $U(2)$, which is reductive but not semi-simple). Then

$$-\det \vec{x} \;=\; t^2 \,-\, x^2 \,-\, y^2 \,-\, z^2 \tag{42}$$

is the Lorenzian norm-square. The Lorenzian inner product is therefore

$$(\vec{x},\, \vec{y}) \;=\; -\frac{1}{2}\left(\det(\vec{x}+\vec{y}) \,-\, \det(\vec{x}) \,-\, \det(\vec{y})\right). \tag{43}$$

Now $SL(2,\mathbb{C})$ acts on the set of anti-hermitian matrices via the conjugation isomorphism

$$Conj : SL(2,\mathbb{C}) \to SO^{+}(1,3), \qquad Conj_M(\vec{x}) \;=\; M\vec{x}\overline{M}^T. \tag{44}$$

This is clearly norm-preserving, although $Conj_M$ is generally not an algebra homomorphism. It is easy to see that $ker(Conj) = \{\pm Id\}$, so this is again a 2-1 covering map. Further, $SL(2,\mathbb{C})$ is simply-connected, so

$$SL(2,\mathbb{C}) \;\approx\; Spin(1,3) \tag{45}$$

and we have found another spin group.