Lecture 10 - F-structures IV - Collapse Implies Existence of an F-structure

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Theorem 0.1 *Given a manifold M^n, there is a decomposition $M^n = K^n \cup H^n$ where H admits an F-structure of positive rank and if $p \in K$, then there is a $c = c(n) < \infty$ such that*

$$\sup_{y \in B_{ci_p}(p)} |\mathrm{Rm}_y|^{1/2} i_p \ \geq \ c^{-1}.$$

We first try to explain the idea behind the proof. The constant $c(n)$ is chosen so that $\sup_{y \in B_{ci_p}(p)} |\mathrm{Rm}_y|^{1/2} i_p < c^{-1}$ implies $B_{ci_p}(p)$ is *almost flat* in the sense that there exists a quasi-isometry from some flat manifold into some large subset (compared to the injectivity radius) of $B_{ci_p}(p)$. We construct (elementary) F-structures on flat manifolds, which then pass to these almost-flat balls. A technical argument remains on how to "glue" the F-structures together on overlaps. This is achieved by showing that the F-structures' local actions are "almost" the same, in the C^1-sense. Then a stability theorem is used: if a Lie group has two actions that are "close enough" in the C^1-sense, the actions can be perturbed so as to coincide.

Essentially the orbits of the F-structure correspond to the "most collapsed directions."

1 Locally collapsed regions

Given $y \in M$ and $R > 0$ define the quantity $v(y, R)$ by

$$v(y, S) \triangleq \sup_{x \in B_{Si_y}(y)} |\mathrm{Rm}_x|^{\frac{1}{2}} i_y.$$

By an *h-quasi-isometry* (for $h \in [1, \infty)$) between Riemannian manifolds U and V will mean a homeomorphism $f : U \to V$ differentiable of degree at least $C^{k,\alpha}$, so that $\frac{1}{h} g_U \ \leq \ f^* g_V \ \leq \ h g_U$. Of course a 1-quasi-isometry is an isometry.

Lemma 1.1 *Given $h > 0$, $k < \infty$, there is a $\delta = \delta(h,k,n)$ and an $R = R(h,k,n)$ so that if $v(y, \delta^{-1}) < \delta$ then there is a flat manifold F with soul S so that*

i) an h-quasi-isometry $f : U \to U_F$ from some subset $y \in U \subset B_{ki_y}(y)$ a neighborhood $U_F \subset F$, where also U contains $B_{\frac{1}{4}ki_y}(y)$,

ii) $\mathrm{dist}(f(y), S) \leq R$,

iii) $\mathrm{Diam}(S) \leq R$.

Pf

Assume (i) is false. Put $\delta_i = i^{-1}$. By scale invariance we can assume that $i_y = 1$ and $|\,\mathrm{Rm}\,| < 1/i$ on $B_i(y)$, but there is no h-quasi-isometry from any neighborhood of y to any tubular neighborhood $B_{i \cdot i_y}(S)$ of any soul in any flat manifold.

But by Cheeger-Gromov convergence, as $i \to \infty$ the sets $B_i(y)$ converge in the $C^{1,\alpha}$-topology to a complete flat manifold with unit injectivity radius at a point.

Thus for large enough i, there is indeed an h-quasi-isometry from $B_i(y)$ to a subset of this flat manifold.

If (ii) or (iii) is false, we can repeat the argument. However, in the limiting flat manifold the soul is a finite distance away, so it is clear that we can chose a subset $U_i \subset B_i(y)$ with $y \in U_i$ that maps onto some tubular neighborhood. □

The h-quasi-isometry is actually too weak a notion. It is important that holonomies converge, not just distances. However since the convergence above occurs in the $C^{1,\alpha}$-topology (in particular, in the C^1 topology), holonomies around geodesic loops based at y converge to the respective holonomies in the flat case.

2 Joining of locally defined F-structures

In this section we look at how F-structures are defined locally, and how they are joined together. Pick $h > 0$. Let $p \in M$ and suppose curvature satisfies $|\,\mathrm{Rm}\,| < \delta\, i_p^{-2}$ inside $B_{i_p \delta^{-1}}(p)$. Then there is some flat manifold, Y_p, and an h-quasi-isometry between a some large subset of $U_p \subset B_{i_p \delta^{-1}}(p)$ and a large subset of Y_p.

There is an F-structure on Y_p, however we do not want the entire F-structure. We will consider a loop at p to be a "short loop" if it is a geodesic lasso and its length is a definite multiple of the injectivity radius. Corresponding to short geodesic loops at p are short almost-geodesic loops in Y_p, which can be homotoped to short (nontrivial!) geodesic loops. If the loops at p have small holonomy, then (by Bieberbach's theorem) the corresponding loops in Y_p have zero holonomy and therefore correspond to geodesic loops in the covering torus, so correspond to an orbit of the F-structure. Let $\gamma_1, \ldots, \gamma_k$ be the loops at p with small holonomy (say, maximal rotation angle $< 1/4$); a simple argument shows this list is nontrivial. Corresponding to these are loops $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ in Y_p, corresponding to which is an F-structure of constant rank k. It is this F-structure which passes down to a neighborhood near p.

Now consider two nearby points p, q with overlapping neighborhoods U_p, U_q. Let $\gamma_1^p, \dots, \gamma_k^p$ and $\gamma_1^q, \dots, \gamma_l^q$ be the short loops at p, q, respectively, with maximal holonomy angle $< \frac{1}{4}$; these lead to possibly different F-structures on $U_p \cap U_q$, although $U_p \cap U_q$ is saturated for either structure.

We claim is that a third structure exists on a neighborhood of $U_p \cap U_q$, which contains both previous structures. One can "slide" the loops $\gamma_1^p, \dots, \gamma_k^p$ and $\gamma_1^q, \dots, \gamma_l^q$ to a point $p' \in U_p \cap U_q$. At p' these loops still have small holonomy and short length, so define an F-structure on a neighborhood of p'.

Now we can replace U_p with $U_p - \overline{U_q}$ and the same with U_p. Repeating this process, we get at least one F-structure defined in a neighborhood of each point, so that if two such structures overlap, then one contains the other.

If $|\operatorname{Rm}|^{1/2} i_x$ is small enough, the orbits of the F-structures will converge in the C^1 sense. A stability theorem (Grove-Karcher (1973)) says that if two Lie groups produce actions that are close enough in the C^1-sense, the actions can be perturbed so as to coincide.