Lecture 2 - Basic Concepts II - Lie Groups

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1 Differential Topology

The basic objects of differential topology are manifolds and diffeomorphisms, which we shall assume familiarity with, including the concepts of differentiability, tangent spaces, tensors, forms, and so on. One purely topological notion that will be useful in the future is that of the Lie derivative, which is a differentiable (not Riemannian) version of taking a derivative with respect to a vector field.

The Lie derivative of a tensor T is defined by taking the derivative of the pullbacks of T along some smooth family of diffeomorphisms. Let X be a vector field and φ_t its flow. This means that

$$X_p(f) = \frac{d}{ds}\Big|_{s=t} f(\varphi_s(\varphi_{-t}(p))). \tag{1}$$

for any $t \in \mathbb{R}$. If $T \in \bigotimes^{p,0} TM$ then we define

$$L_X T \quad = \quad -\frac{d}{dt}\varphi_{t*}T \tag{2}$$

where $\varphi_{t*}T$ indicates the push-forward of T along the flow. If $T \in \bigotimes^{0,q} TM$ then

$$L_X T \quad = \quad \frac{d}{dt}\varphi_t^* T \tag{3}$$

In the general case where $T \in \bigotimes^{p,q} TM$, for instance if $T = Y_1 \otimes \cdots \otimes Y_p \otimes \eta^1 \otimes \cdots \otimes \eta^q$ is a simple tensor, we define

$$L_X T \;=\; \lim_{t\to 0} \frac{Y_1 \otimes \cdots \otimes Y_p \otimes (\varphi_t^*\eta^1) \otimes \cdots \otimes (\varphi_t^*\eta^q) \;-\; (\varphi_{t*}Y_1) \otimes \cdots \otimes (\varphi_{t*}Y_p) \otimes \eta^1 \otimes \cdots \otimes \eta^q}{t} \tag{4}$$

and extend by linearity.

The definition of the Lie derivatuve involves no concepts except diffeomorphisms; the Lie derivative is a concept of differential topology, not geometry. If X, Y are vector fields, f

is a function, and η is any p-form, and T is any tensor, then the following extremely useful facts can be shown:

i) $L_X f = X(f)$

ii) $L_X Y = [X, Y]$

iii) $[L_X,\, d]\,\eta \;=\; 0$

iv) $[L_X,\, i_Y]\,\eta \;=\; i_{[X,Y]}\eta$

v) Cartan's formula: $L_X\eta \;=\; [d, i_x]\eta$

vi) The Jacobi identity: $L_{[X,Y]}T \;=\; [L_X, L_Y]T$

If A and B are operators on the exterior algebra $\bigwedge^* TM$ of degrees $|A|$ and $|B|$, respectively, we use the sign convention

$$[A,\, B] \quad = \quad AB \,-\, (-1)^{|A||B|}BA \tag{5}$$

Note that the Lie derivative L_X has degree zero, i_X has degree -1, and d has degree $+1$, so for instance in Cartan's formula we have $[d, i_X] = di_x + i_x d$.

It is also possible to define the Lie derivative using its properties, as was done for the covariant derivative. Given a vector field X, then $L_X : \bigotimes^{p,q} TM \to \bigotimes^{p,q} TM$ is the unique operator with the following properties:

I) $L_X T$ is linear in the X-variable

II) $L_X T$ is linear in the T-variable

III) $L_X f = X(f)$

IV) L_X obeys the Leibnitz rule with respect to both tensor products and contractions

V) If Y is a vector field, then $L_X Y = [X, Y]$.

The Lie derivative is not a connection, since tensorality is not present in the first variable.

2 Lie Groups

A Lie group G is a topological group with a differentiable structure, and so that the group operations are diffeomorphisms. Let $a \in G$ and let L_a be left-multiplication by a and R_a be right-multiplication by a. Also let I be the inverse mapping: $I(a) = a^{-1}$. Then the operators

$$R_a : G \to G, \quad L_a : G \to G, \quad I : G \to G \tag{6}$$

are required to be diffeomorphisms for all $a \in G$. If X is a vector field on G, then it is said to be *left-invariant* if

$$(L_a)_* X_p \;=\; X_{ap} \tag{7}$$

for all $a, p \in G$. The similar definition for right-invariance is obvious. Thus if X is any left-invariant field, it is determined by its value at a single point; one usually specifies a left-invariant field simply by specifying its value at the identity element $e \in G$.

The operator $Ad_a : G \to G$ acts on G by

$$Ad_a = L_a \circ R_{a^{-1}}. \tag{8}$$

Of course this takes e to e, but it is not the identity on G unless $a \in Z(G)$. The derivative of Ad_a, also (unfortunately) denoted Ad_a, operates on vector fields by

$$Ad_a = (L_a)_*(R_{a^{-1}})_*. \tag{9}$$

Note that $Ad_a : T_eM \to T_eM$ so Ad is a representation $Ad : G \to GL(T_eG)$ of G into the group of nonzero transformation of T_eM. When restricted to left-invariant fields, Ad_a is entirely determined by its action on T_eG.

The exponential map $\mathrm{Exp} : T_eM \to G$ is defined as follows. If $X \in T_eG$ then X can be regarded as a left-invariant field on G. If $t \in \mathbb{R}$, then $\mathrm{Exp}(tX)$ is the point in G obtained by following the integral curve of X to a distance of t.

If G is a matrix group, then T_eG is canonically a vector space of matrices, and it can be proven that $\mathrm{Exp}(tX)$ is the exponential series

$$\mathrm{Exp}(tX) \;=\; I + tX + \frac{1}{2}t^2X^2 + \ldots \tag{10}$$

Additionally, if G is a matrix group and $a(t) = \mathrm{Exp}(tX)$, then $a(t)^{-1} = a(-t)$ and

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} Ad_{a(t)}Y &= \left(\frac{d}{dt}\Big|_{t=0} L_{a(t)}\right) Y + \left(\frac{d}{dt}\Big|_{t=0} R_{a(t)^{-1}}\right) Y \\ &= \left(\frac{d}{dt}\Big|_{t=0} a(t)\right) Y + Y\left(\frac{d}{dt}\Big|_{t=0} a(-t)\right) \\ &= XY - YX. \end{aligned} \tag{11}$$

We therefore define

$$ad_X Y \;\triangleq\; \frac{d}{dt} Ad_{a(t)}Y \tag{12}$$

where $a(t) = \mathrm{Exp}(tX)$. We often write $ad_X Y = [X, Y]$. It is easily proven that $[X, Y] = -[Y, X]$ and that the Jacobi identity holds, so T_eM has the structure of a Lie algebra.

Example: The half-line.

The group of positive real numbers forms a Lie group under multiplication, with 1 being the identity element. This Lie group is of course abelian. We have $T_eG \approx \mathbb{R}$, and if $X \in \mathbb{R} \approx T_eG$ then $\mathrm{Exp}(tX) = e^{tX}$.

<u>Example: $\mathbb{S}^1$</u>

$\mathbb{S}^1$ has the structure of a Lie group when it is regarded as the set $\{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$, along with complex multiplication. The identity is again $1 = e^0$, and the tangent space at 1 is naturally identified with the pure-imaginary numbers. The exponential map $\mathrm{Exp} : i\,\mathbb{R} \to \mathbb{S}^1$ is, again, simply $\mathrm{Exp}(iX) = e^{iX}$.

<u>Example $\mathbb{S}^3$</u>

The Lie group structure of $\mathbb{S}^3$ can be seen in several ways. In analogy with the case of $\mathbb{S}^1$, which we defined to be the unit complex numbers, we can define $\mathbb{S}^3$ to be the unit quaternions, namely the set $\{\, q \in \mathbb{H} \mid |q|^2 = 1 \,\}$, where $\mathbb{H}$ is the quaternion algebra. Since quaternion-multiplication is norm-preserving, meaning

$$|q_1q_2|^2 \quad = \quad |q_1|^2|q_2|^2 \tag{13}$$

we see that quaternionic multiplication descends to $\mathbb{S}^3$, giving it a non-abelian Lie group structure. The identity element e is $1 \in \mathbb{H}$, and the vector space $T_e\,\mathbb{S}^3$ is naturally identified with the vector space of purely imaginary quaternions.

If $\mathbf{v} \in T_e\,\mathbb{S}^3$, meaning $\mathbf{v} = a\,i + b\,j + c\,k$, we have again that $\mathrm{Exp}(t\mathbf{v}) = e^{t\mathbf{v}}$. If $\mathbf{v}_1, \mathbf{v}_2 \in T_e\,\mathbb{S}^3$ are any purely imaginary quaternions, then by taking derivatives it is easy to see that the Lie algebra is simply

$$[\mathbf{v}_1, \mathbf{v}_2] \;=\; \mathbf{v}_1\mathbf{v}_2 \;-\; \mathbf{v}_2\mathbf{v}_1 \tag{14}$$

where the product, for example $\mathbf{v}_1\mathbf{v}_2$, is just quaternionic multiplication. Computed in the natural basis, this means

$$[i,\, j] \;=\; 2k, \quad [j,\, k] \;=\; 2i, \quad [k,\, i] \;=\; 2j, \tag{15}$$

so that T_eM is the Lie algebra $\mathfrak{so}(3)$ (with the usual basis elements multiplied by 2). This is of course the same as the Lie algebra $\mathfrak{su}(2)$.

It is possible to see that $\mathbb{S}^3$ is the Lie group $SU(2)$, not $SO(3)$. Note that a homomorphism $\mathbb{H} \to \mathbb{C}(2)$ exists ($\mathbb{C}(2)$ is the group of complex 2×2 matrices), which sends $q = a + bi + cj + dk$ to the matrix

$$M_q \;=\; \left[\begin{array}{cc} a - di & -c - bi \\ c - bi & a + di \end{array}\right]. \tag{16}$$

There are in fact many ways to choose this homomorphism, although up to conjugation by group elements there are only two ways, which are related to each other by quaternionic conjugation.

Note that $|q|^2 = \det(M_q)$. Since $q \in \mathbb{S}^3$ implies $|q|^2 = 1$, we have proven that $q \mapsto M_q$ maps $\mathbb{S}^3$ to $SL(2)$. Any matrix of the form M_q above is also unitary (see Exercise 7), so that in fact $\mathbb{S}^3$ maps to $SU(2)$. The kernel is easily seen to be trivial, and it can be proven that this is surjective as well. Therefore $\mathbb{S}^3 \approx SU(2)$.

This is a Lie groups isomorphism, so the Lie algebras should be canonically isomorphic as well. To check this, let γ_i, γ_j, and γ_k be paths in $\mathbb{S}^3$ that represent the vectors i, j, and k at the point $1 \in \mathbb{S}^3$. A natural choice is

$$\begin{aligned} \gamma_i(t) &= \cos(t) + i\sin(t) \\ \gamma_j(t) &= \cos(t) + j\sin(t) \\ \gamma_k(t) &= \cos(t) + k\sin(t). \end{aligned} \tag{17}$$

Under the mapping $q \mapsto M_q$, we obtain the three paths through Id

$$\begin{aligned} \Gamma_x(t) &= \begin{bmatrix} \cos(t) & -i\sin(t) \\ -i\sin(t) & \cos(t) \end{bmatrix} \\ \Gamma_y(t) &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \\ \Gamma_z(t) &= \begin{bmatrix} \cos(t) - i\sin(t) & 0 \\ 0 & \cos(t) + i\sin(t) \end{bmatrix} \end{aligned} \tag{18}$$

so that the tangent space is

$$T_{Id}SU(2) = \left\{ \begin{bmatrix} -zi & -y - xi \\ y - xi & zi \end{bmatrix} \;\middle|\; x,\, y,\, x \in \mathbb{R} \right\}. \tag{19}$$

This is $\mathfrak{su}(2)$, the Lie algebra of *trace-free anti-Hermitian matrices*. A basis over $\mathbb{R}$ is $-i$ times the usual Pauli matrices σ_x, σ_y, and σ_z, where

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{20}$$

We have shown

$$\begin{aligned} &T_{Id}SU(2) \approx \mathrm{span}_{\mathbb{R}}\{-i\sigma_x,\, -i\sigma_y,\, -i\sigma_z\} \qquad \text{and} \\ &[-i\sigma_x,\, -i\sigma_y] = -2i\sigma_z \qquad [-i\sigma_y,\, -i\sigma_z] = -2i\sigma_x \qquad [-i\sigma_z,\, -i\sigma_x] = -2i\sigma_y \end{aligned} \tag{21}$$

The Lie algebra structure given by (21) is of course identical to that given by (15).

The Pauli matrices first appeared in the study of electron spin, and continue to be of vital importance in physics. The reason these spin matrices should appear in the study of $\mathbb{S}^3$ may at first seem strange. However $\mathbb{S}^3$ is actually a spin group, $Spin(3)$, the double cover of $SO(3)$.

3 Exercises

1) Let G be a group. Prove that the bracket operation on T_eG defined by (12) and the topological bracket of the corresponding left-invariant fields coincide.

2) Let G be a Lie group. Prove that the topological bracket between any left-invariant field and any right-invariant field on G is zero.

3) Prove that the differential of the adjoint action Ad_a on a Lie group takes left-invariant vector fields to left-invariant vector fields, and right-invariant fields to right-invariant fields.

4) Give an explicit proof of (14).

5) It was asserted that the homomorphism (16) could be chosen differently. What are some different choices?

6) Prove that, under the mapping $q \mapsto M_q$, conjugation by the group element j (that is, the map $q \mapsto jqj^{-1}$) corresponds to the matrix complex conjugation operation.

7) Prove that, under the mapping $q \mapsto M_q$, quaternionic conjugation corresponds to the matrix conjugate-transpose operation. Use this to prove that $\mathbb{S}^2 \to U(2)$.

8) Prove that any 2×2 unitary matrix (defined to be a complex matrix U with $U\overline{U}^T = Id$) has the form $e^{i\theta/2}M_q$ for some unit quaternion q. Use this to prove that $\mathbb{S}^3 \approx SU(2)$.

9) Prove that the homomorphism (16) is unique up to quaternionic conjugation (the map $a+bi+cj+dk \mapsto a-bi-cj-dk$) and conjugation by group elements (that is, mappings of the form $q \mapsto aqa^{-1}$ where a is another quaternion).

10) Prove that the 4-dimensional algebra of 2×2 anti-hermitian matrices constitute the Lie algebra of $U(2)$.