Lecture 3 - Lie Groups and Geometry

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1 Integration of Vector Fields on Lie Groups

Let M be a complete manifold, with a vector field X. A time-dependent family of diffeomorphisms $\varphi_t : M \to M$ is called the flow of X if, for any function f and any time τ, we have

$$\frac{d}{dt}\Big|_{t=\tau} f\left(\varphi_t\left(\varphi_\tau^{-1}(p)\right)\right) \quad = \quad X_p(f) \tag{1}$$

The path $t \mapsto \varphi_t(p)$ is called the *flow line*, or *integral curve*, of X through p.

Now let G be a Lie group, and assume X is a left-invariant vector field. The flow line of X through e is, by definition, the path $t \mapsto \mathrm{Exp}(tX)$. Moving this by left-translation to any point $a \in G$, the flow line through a is

$$t \mapsto L_a \,\mathrm{Exp}(tX).$$

Thus the flow of X is therefore

$$\begin{aligned} \varphi_t(a) &= a\,\mathrm{Exp}(tX) \\ &= R_{\mathrm{Exp}(tX)}a. \end{aligned}$$

Thus $\varphi_t = R_{\mathrm{Exp}(tX)}$, so that left-invariant vector fields integrate out to smooth families of *right* translations. Similarly right-invariant fields integrate out to left translations.

2 Riemannian Geometry on Lie Groups

2.1 Left-invariant metrics

If g_e is an inner product on the tangent space at the identity, we can extend g_e to a metric g on G by left translation. That is, if $a \in G$ and X_a, Y_a are vectors at a, then

$$g(X_a, Y_a) \quad \triangleq \quad g(L_{a^{-1}*}X_a,\, L_{a^{-1}*}Y_a). \tag{2}$$

Note that a left-invariant metric g is constant when evaluated against left-invariant fields Y and Z. Thus

$$(L_X g)(Y, Z) = -g([X, Y], Z) - g(Y, [X, Z]) \tag{3}$$

Note that $[X, W] = 0$ whenever X is right-invariant and W is left-invariant; this is from an exercise from a previous lecture. When g is a left-invariant metric, then *right*-invariant fields are Killing.

2.2 Bi-invariant metrics

A metric is called *bi-invariant* if it is both left- and right-invariant. If X is a left-invariant field and g is a right-invariant metric, then $L_X g = 0$. Therefore if g is bi-invariant and X, Y, Z are left-invariant, then

$$0 = g([X, Y], Z) + g(Y, [X, Z]). \tag{4}$$

Most Lie groups do not have bi-invariant metrics, although all compact Lie groups do. From (3) it is easy to see that semi-simple Lie groups have a bi-invariant metric, as the Killing form automatically satisfies (4) and is definite on semi-simple algebras.

2.3 Connection and curvature in a bi-invariant metric

If a Lie group has a bi-invariant metric, the Lie algebra entirely determines the Riemannian structure. If g is left-invariant and $X_1, \ldots, X_n$ are left-invariant fields that form an orthonormal basis at e (and therefore at all points), the Koszul formula reduces to

$$2\langle \nabla_{X_i} X_j, X_k \rangle = \langle [X_i, X_j], X_k \rangle - \langle [X_j, X_k], X_i \rangle + \langle [X_k, X_i], X_j \rangle$$

If g is bi-invariant, by (4) the last 2 terms cancel and we get $2\langle \nabla_{X_i} X_j, X_k \rangle = \langle [X_i, X_j], X_k \rangle$, so that

$$\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j]. \tag{5}$$

Using the Jacobi identity and (4), we easily find

$$\begin{aligned} \mathrm{Rm}(X, Y)Z &= -\frac{1}{4}[[X, Y], Z] \\ \langle \mathrm{Rm}(X, Y)Z, W \rangle &= \frac{1}{4}\langle [X, Y], [W, Z] \rangle \\ sec(X, Y) &= \frac{1}{4}\frac{|[X, Y]|^2}{|X|^2|Y|^2}. \end{aligned} \tag{6}$$

Sectional curvatures are therefore all non-negative.

3 Examples

3.1 The 3-sphere

Again the fundamental example is the 3-sphere. We have seen that $\mathbb{S}^3$ is a Lie group in two equivalent ways, as the group of unit quaternions, and as the matrix group $SU(2)$. We have seen that its Lie algebra has two equivalent formulations, as the Lie algebra of purely imaginary quaternions, and as the span of $-i$ times the Pauli matrices.

In the metric inherited from $\mathbb{R}^4$, the tangent vectors $\{i, j, k\} \in T_1\,\mathbb{S}^3$ are orthonormal. It is a simple matter to verify that the inherited metric is bi-invariant. Letting $\mathbf{v}_i$, $\mathbf{v}_j$, and $\mathbf{v}_k$ be the left-invariant fields on $\mathbb{S}^3$ with $\mathbf{v}_i(1) = i$, $\mathbf{v}_j(1) = j$ and $\mathbf{v}_k(1) = k$, then with the Lie algebra structure $[\mathbf{v}_i, \mathbf{v}_j] = 2\mathbf{v}_k$ and cyclic permutations, we compute the connections

$$\begin{aligned} \nabla_{\mathbf{v}_i}\mathbf{v}_i \;&=\; \nabla_{\mathbf{v}_j}\mathbf{v}_j \;=\; \nabla_{\mathbf{v}_k}\mathbf{v}_k \;=\; 0 \\ \nabla_{\mathbf{v}_i}\mathbf{v}_j \;&=\; \mathbf{v}_k \\ \nabla_{\mathbf{v}_j}\mathbf{v}_k \;&=\; \mathbf{v}_i \\ \nabla_{\mathbf{v}_k}\mathbf{v}_i \;&=\; \mathbf{v}_j, \end{aligned} \tag{7}$$

the curvatures

$$\begin{aligned} \mathrm{Rm}(\mathbf{v}_i,\,\mathbf{v}_j)\mathbf{v}_k \;&=\; 0 \\ \mathrm{Rm}(\mathbf{v}_i,\,\mathbf{v}_j)\mathbf{v}_j \;&=\; \mathbf{v}_i \\ &\mathit{etc.}, \end{aligned} \tag{8}$$

and the sectional curvatures

$$sec(\mathbf{v}_i,\,\mathbf{v}_j) \;=\; 1, \qquad sec(\mathbf{v}_j,\,\mathbf{v}_k) \;=\; 1, \qquad sec(\mathbf{v}_i,\,\mathbf{v}_k) = 1. \tag{9}$$

As we already know to be the case, $\mathbb{S}^3$ has constant curvature $+1$.

3.2 The Hopf Fibration

Note that $\mathbb{H}$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, for instance by writing $q = a + bi + cj + dk$ as

$$q \;\;=\;\; (a\,+\,bi)\,+\,(c\,+\,di)j. \tag{10}$$

Let X_L, Y_L, and Z_L be the vector fields obtained by left-translating the vectors $i, j, k \in T_1\,\mathbb{S}^3$. The corresponding integral curves through 1 are $t \mapsto e^{it}$, $t \mapsto e^{jt}$, and $t \mapsto e^{kt}$. Left-translating to any $q \in \mathbb{S}^3$, the flow $\varphi_t : \mathbb{S}^3 \to \mathbb{S}^3$ of X_L is

$$\varphi_t \;=\; R_{e^{it}}. \tag{11}$$

Through a point $q = a + bi + cj + dk$, this is

$$\varphi_t(q) \;=\; q\,e^{it} \;=\; e^{it}(a + bi) \,+\, e^{-it}(c + di)j. \tag{12}$$

Viewing this via stereographic projection from -1, it is easy to see that this is a left-handed (clockwise) screw motion (assuming $\mathbb{S}^3$ is oriented with the outward normal).

If i is extended instead to a right-invariant field X_R, then the flow is

$$q \mapsto e^{it} q \;=\; e^{it}(a+bi) \,+\, e^{it}(c+di)j \tag{13}$$

which is a flow by left-translations, and a right-handed (counterclockwise) screw motion.

3.3 Nilpotent groups

The set of upper-triangular $n \times n$ matrices with 1's along the diagonal is a nilpotent group under matrix multiplication. From its importance in physics, the group of upper-triangular 3×3 matrices

$$\mathcal{H} \;=\; \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x,\, y,\, z \,\in\, \mathbb{R} \right\} \tag{14}$$

is given the name *the Heisenberg group*. Of course it it topologically just $\mathbb{R}^3$. Its Lie algebra consists of the strictly upper-triangular matrices

$$\mathfrak{h} \;=\; \mathrm{span}_{\mathbb{R}} \left\{ P \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\; Q \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\; h \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \tag{15}$$

where we see

$$[P,\, Q] \;=\; h, \quad [P,\, h] \;=\; 0, \quad [Q,\, h] \;=\; 0. \tag{16}$$

With the left-invariant metric, the Koszul formula gives

$$\begin{array}{lll} \nabla_P P \;=\; 0 & \nabla_Q P \;=\; -\frac{1}{2}h & \nabla_h P \;=\; -\frac{1}{2}Q \\ \nabla_P Q \;=\; \frac{1}{2}h & \nabla_Q Q \;=\; 0 & \nabla_h Q \;=\; \frac{1}{2}P \\ \nabla_P h \;=\; -\frac{1}{2}Q & \nabla_Q h \;=\; \frac{1}{2}P & \nabla_h h \;=\; 0 \end{array} \tag{17}$$

and we compute

$$\begin{array}{ll} \mathrm{Rm}(P,Q)Q \;=\; -\frac{3}{4}P & \mathrm{Rm}(P,h)h \;=\; \frac{1}{4}P \\ \mathrm{Rm}(Q,h)h \;=\; \frac{1}{4}Q & \mathrm{Rm}(Q,P)P \;=\; -\frac{3}{4}Q \\ \mathrm{Rm}(h,P)P \;=\; \frac{1}{4}h & \mathrm{Rm}(h,Q)Q \;=\; \frac{1}{4}h \end{array} \tag{18}$$

with all other curvatures, such as $\mathrm{Rm}(P,Q)h$, being zero. Therefore

$$sec(P,Q) \;=\; -\frac{3}{4}, \quad sec(P,h) \;=\; \frac{1}{4}, \quad sec(Q,h) \;=\; \frac{1}{4}, \tag{19}$$

so $\mathcal{H}$ has both positive and negative sectional curvatures. We easily compute

$$\mathrm{Ric} \;=\; \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{20}$$

so scalar curvature is negative: $R = -\frac{1}{2}$.

4 Exercises

1) Let G be a connected Lie group with a bi-invariant metric, and Lie algebra $\mathfrak{g}$. Prove that the representation $Ad : G \to GL(\mathfrak{g})$ reduces to a representation $Ad : G \to SO(T_e\, \mathfrak{g})$.

2) Identifying the Heisenberg group $\mathcal{H}$ with $\mathbb{R}^3$, what are expressions (in (x, y, z)-coordinates) of the left- and right-invariant vector fields that extend P, Q, and h?

3) Is the metric on $\mathcal{H}$ that was given in the text a bi-invariant metric?

4) Let V be the infinite-dimensional vector space of polynomials in x. Let λ be a constant. Prove that sending P to the operator $\lambda \frac{d}{dx}$, Q to the operator that multiplies by x, and h to the operator that multiplies by λ, is a representation of $\mathcal{H}$ on V (in physics normally $\lambda = i\hbar$).

5) The simplest non-trivial Lie algebra has two generators A and B with $[A, B] = B$. Obviously it is nilpotent. What is the corresponding Lie Group G? Given the left-invariant metric with $|A| = |B| = 1$ and $\langle A, B\rangle = 0$, what is the curvature? Prove that G has no bi-invariant metric.