Lecture 8 - Flat Manifolds and Invariant F-structures

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1 An F-structure with no Polarized Substructures

1.1 The spaces $\mathcal{E}_\theta$

Let $\mathcal{E} = \mathbb{R}\times\mathbb{C}$ and define $\mathcal{E}_\theta = \mathcal{E}\,/\Gamma$, where Γ is the infinite cyclic group on one generator γ which sends $(x,v)\mapsto(x+2\pi, ve^{i\theta})$. Note that $\mathcal{E}_\theta$ is diffeomorphic to $\mathbb{S}^1\times\mathbb{C}$, although it has a different geometric structure. Geometrically, we have $\mathcal{E}_0 = \mathcal{E}_{2\pi}$.

Define an action of $\mathbb{R}\times\mathbb{S}^1$ on $\mathcal{E}$ as follows. If $(t, e^{is})\in\mathbb{R}\times\mathbb{S}^1$ then define

$$(t,\, e^{is}).(x,\, v) \;=\; (x+t,\, ve^{i(s+t\frac{\theta}{2\pi})}). \tag{1}$$

Defining $\gamma:\mathbb{R}\times\mathbb{S}^1\to\mathbb{R}\times\mathbb{S}^1$ by $\gamma.(t,e^{is}) = (t+2\pi, e^{is})$, then note that if $n,k\in\mathbb{Z}$ we have

$$\begin{aligned}\left(\gamma^n.(t,\,e^{is})\right).\left(\gamma^k.(x,\,v)\right) &= (t+2\pi n,\, e^{is}).(x+2\pi k,\, ve^{ik\theta})\\ &= \left(x+t+2\pi(n+k),\, ve^{i(s+t\frac{\theta}{2\pi}+(n+k)\theta)}\right)\\ &= \gamma^{n+k}.(x+t,\, ve^{is+t\frac{\theta}{2\pi}})\\ &= \gamma^{n+k}.\left((t,e^{is}).(x,\,v)\right)\end{aligned} \tag{2}$$

Therefore the action of $\mathbb{R}\times\mathbb{S}^1$ on $\mathcal{E}$ passes to both quotients, becoming an action of T^2 on $\mathcal{E}_\theta$. This gives a pure T-structure of nonconstant rank.

1.2 An F-structure with no polarization

Now consider the space X diffeomorphic to $[0,1]\times\mathbb{S}^1\times\mathbb{C}$, but give each $\{\theta\}\times\mathbb{S}^1\times\mathbb{C}$ the metric structure of $\mathcal{E}_\theta$, and assign it the torus action given above for $\mathcal{E}_\theta$. Since $\mathcal{E}_0$ is isometric to $\mathcal{E}_1$, we can identify $\{0\}\times\mathbb{S}^1\times\mathbb{R}^2$ and $\{1\}\times\mathbb{S}^1\times\mathbb{R}^2$. We will call the resulting space M^4.

If $f : \mathcal{E}_{2\pi} \to \mathcal{E}_0$ is the canonical identification, the torus action on $\mathcal{E}_{2\pi}$ pushes forward to $\mathcal{E}_0$ via

$$f_*(t,\, e^{is}).(x,v) \;=\; (x+t,\, e^{i(s+t)}) \;=\; (t, e^{i(s+t)}).(x,v) \tag{3}$$

This produces the torus map $(t, e^{is}) \mapsto (t, e^{i(s+t)})$, which in terms of the parameters $\{t,s\}$ is given by the map

$$\begin{pmatrix} t \\ s \end{pmatrix} \quad\mapsto\quad \begin{pmatrix} t \\ t\,+\,s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} \tag{4}$$

The space X is diffeomorphic to the 4-manifold $T^2 \times \mathbb{C}$. However, if one travels along the latitude of the base T^2, the stalks of the F-structure (which are also 2-tori) have holonomy given by the matrix

$$A \;=\; \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \;\in\; SL(2,\, \mathbb{Z}), \tag{5}$$

where A is regarded as an automorphism of the torus. Let $\mathcal{G}$ denote the T-structure obtained on M^4 obtained by gluing the torus actions on $\mathcal{E}_{2\pi}$ to that of $\mathcal{E}_0$ by the matrix A.

Let $S^2 \subset M^4$ be the central 2-torus. That is, S is the set of points p so that if $p \in \mathcal{E}_t$ then p has the form $(t,0)$. The matrix A has a single eigenvalue, which corresponds to the rotation that fixes the the base circle in each $\mathcal{E}_\theta$. Therefore the only pure substructure of $\mathcal{G}$ corresponds to the circle action that fixes S, and therefore has zero rank at points of S^2. This proves that M^4 has no pure substructure of non-zero rank on *any* neighborhood of S^2.

Since $\mathcal{G}$ admits no pure substructure of positive rank on a neighborhood of S^2, it admits no polarized substructure.

2 Elementary F-structures on Complete Flat Manifolds

Let X^n be a complete flat manifold. By the splitting theorem, $X^n = \overline{X}^k \times \mathbb{R}^{n-k}$, where $\overline{X}^k$ has no lines. By the Soul theorem, $\overline{X}^k$ has the structure of a normal bundle over a compact, totally geodesic submanifold S^l, called to soul. The choice of a soul in $\overline{X}^k$ is unique, and since it is totally geodesic, it is flat. By the Bieberbach theorem, $S^l \approx T^l/\Gamma$, where T^l is the l-torus and Γ is a discrete group of automorphisms of T^l with $|\Gamma| \leq \lambda$, where $\lambda = \lambda(l)$ is a dimensional constant determined by the proof of the Bieberbach theorem (indeed $\lambda(l) \leq 2(4\pi)^{\frac{1}{2}l(l-1)}$).

Set $G = \pi_1(S^l)$. Since $S^l \hookrightarrow X^n$ is a homotopy equivalence, we also have $G \approx \pi_1(X^n)$. The Beiberbach theorem specifically states that a normal, abelian subgroup of finite index $A \subset G$ exists and acts as a group of translations on the universal cover $\mathbb{R}^l$ of S^l.

Now consider the universal cover $\mathbb{R}^n$ of X^n. We denote by E_n the Euclidean group of $\mathbb{R}^n$; of course $E_n = \mathbb{R}^n \ltimes O(n)$. There is a representation $\pi_1(X^n) \hookrightarrow E_n$, which reduces to a representation $A \hookrightarrow E_n$, the target of which is a discrete, commutative subgroup of the Euclidean group. Let $A' \subset A$ be the subgroup whose target consists of orientation-preserving Euclidean motions. Note that A' is still normal in $\pi_1(X^n)$.

The image of a set of generators of A' is a set $\{(w_i, e^{B_i})\}_{i=1}^N$ of commuting elements of E_n. Further, since A' acts as translations on $\mathbb{R}^l$, acts as rotations on an orthogonal subspace $\mathbb{R}^{k-l}$, and acts as the identity on the remaining orthonogal subspace $\mathbb{R}^{n-k}$, we have that $w_i \in \mathbb{R}^l$ and that $e^{B_i} w_i = w_i$.

We can use the generators of the action of A' as generators of an action of $(\mathbb{R}^N, +)$ on $\mathbb{R}^n$, by sending

$$(t_1, \ldots, t_N) \mapsto (t_1 w_1, e^{t_1 B_1}) \circ \ldots \circ (t_N w_N, e^{t_N B_n}).$$

The w_i are translations on $\mathbb{R}^l$ and the e^{B_i} are rotations on the complimentary dimensions, so $w_i e^{B_j} = w_i$. Therefore

$$\begin{aligned} (t_i w_i, e^{t_i B_i}) \circ (t_j w_j, e^{t_j B_j}) &= (t_i w_i + t_j w_j, e^{t_i B_i + t_j B_j}) \\ &= (t_j w_j, e^{t_j B_j}) \circ (t_i w_i, e^{t_i B_i}) \end{aligned}$$

and so this is indeed a group action of $\mathbb{R}^N$ on $\mathbb{R}^n$. Since A' embeds in $\mathbb{R}^N$ as the integer lattice, we get an action of $\mathbb{R}^N / A'$ on $\mathbb{R}^n / A'$. Therefore we have a torus action (not necessarily effective) on $\mathbb{R}^n / A'$. Passing to a quotient of the torus, this is effective. This produces an elementary F-structure on X^n.