Lecture 9 - F-structures III - F-structures Imply Collapse

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1 Pure polarized collapse

Assume the (possibly noncompact) manifold X admits a pure polarized F-structure. This means $\mathfrak{g}$ is a locally constant sheaf whose orbits all have the same dimension. On such a manifold we can split the metric into two parts $g = g' + h$ where h vanishes on vectors tangent to the orbits and g' vanishes on vectors perpendicular to orbits. Set

$$g_\delta = \delta^2 g' + h. \tag{1}$$

Theorem 1.1 (Pure Polarized Collapse) *Assume a manifold M admits a pure polarized F-structure. As $\delta \to 0$, the metric g_δ given by (1) has the following properties. The injectivity radius at any point converges to zero, given any $p, q \in M$, $\mathrm{dist}_{g_\delta}(p, q) = O(\delta)$, and the sectional curvature is uniformly bounded on any compact set.*

<u>*Pf*</u>

We examine the curvature at the point p by constructing special coordinates near p. Let k denote the dimension of the orbits Let N^{n-k} be any submanifold through p transverse to the orbits. Given coordinates $y^1, \dots, y^{n-k}$ on N we can extend these coordinate functions to a neighborhood of p by projecting along the orbits. Finally a k-torus acts locally on the orbits themselves; the push-forward of a basis of its Lie algebra is an independent Abelian set of Killing fields parallel to the orbits, and which span the distribution defined by the orbits. The Frobenius theorem says we can integrate these to get the remaining coordinate functions $x^1, \dots, x^k$ with coordinate fields $\frac{d}{dx^i}$ equal to the original killing fields. We can choose the origin on any orbit to be its point of intersection with N.

The coordinates field $\frac{d}{dy^i} = X_i + V_i$ can be decomposed into a part parallel to the orbits X_i and a part perpendicular to the orbits V_i. Now make the change of coordinates $u^i = \delta x^i$. Then

$$g_\delta = \begin{pmatrix} \left\langle \frac{d}{dx^i}, \frac{d}{dx^j} \right\rangle_{g_1} & \delta \left\langle \frac{d}{dx^i}, \frac{d}{dy^j} \right\rangle_{g_1} \\ \delta \left\langle \frac{d}{dy^i}, \frac{d}{dx^j} \right\rangle_{g_1} & \delta^2 \left\langle X_i, X_j \right\rangle_{g_1} + \left\langle V_i, V_j \right\rangle_{g_1} \end{pmatrix}.$$

As $\delta \to 0$ the metric converges to a warped product metric.

2 Polarized collapse

If the polarization is not pure, it means that the various U_α in the atlas are such that the corresponding pure substructures $\mathcal{G}_\alpha$ possibly have different ranks (though the rank is constant on each U_α). We have to modify the metric on each U_α separately, and at the same time push the various U_α away from each other.

Theorem 2.1 *If $\mathcal{G}$ is a polarized F-structure on the compact manifold X, then X admits a sequence of metrics g_δ so that*

(1) The manifold (X, g_δ) collapses

(2) $\operatorname{diam}_{g_\delta}(X) < \operatorname{diam}_{g_1}(X)|\log\delta|$

(3) $\operatorname{Vol}_{g_\delta}(X) < \operatorname{Vol}_{g_1}(X)\delta^k|\log\delta|^n$*, some* $k \geq 1$

(4) Sectional curvature $|K|$ is uniformly bounded.

<u>*Pf*</u>

Let $\{(U_i, \mathcal{G}_i)\}_{i=1}^N$ be an atlas. Let $f_\alpha : U_\alpha \to [1,2]$ be a collection of functions, constant on the orbits of $\mathcal{G}$, so that $f_i = 1$ in a neighborhood of ∂U_i, and so that $\bigcup_i f_i^{-1}(2) = X$. Put

$$\rho_i \;=\; \delta^{\log_2 f_i}.$$

We start with the metric $g_0 = \log_2(\delta)\, g$. On U_i we can write

$$g_0 \;=\; g_1' \,+\, h_1,$$

where g_1' is tangent to the orbits of $\mathcal{G}_1$ and h_1 is perpendicular. Then define g_1 by

$$g_1 \;=\; \begin{cases} \rho_1^2 g_1' + h_1 & \text{on } U_1 \\ g_0 & \text{on } X \,-\, U_1 \end{cases}$$

Proceed inductively. Once g_{i-1} has been chosen, set $g_{i-1} = g_i' + h_i$ on U_i where g_i' is parallel to the orbits of $\mathcal{G}_i$ and h_i is perpendicular, and put

$$g_i \;=\; \begin{cases} \rho_i^2 g_i' + h_i & \text{on } U_i \\ g_{i-1} & \text{on } X \,-\, U_i \end{cases}$$

Now *(1)*, *(2)*, and *(3)* are obvious, where $k = \min \operatorname{rank} \mathcal{G}_i$.

We check that sectional curvature is bounded. Let $p \in X$; let $l = \dim \mathcal{O}_p$. Let $\{U_j\}_{j=1}^s$ indicate the set of atlas charts in which p lies. If we work on a normal atlas, we have that the ranks l_j of the $\{\mathcal{G}_j\}j = 1^s$ are strictly increasing: $l_1 > \cdots > l_s \geq k$. The metric near p is changed s times, and we will keep track of the changes in curvature as the metric is changed each time.

Let N^{n-l_j} be a submanifold transverse to the orbits of $\mathcal{G}_j$, and choose coordinates $(\underline{x}^1, \ldots, \underline{x}^l, \underline{y}^1, \ldots, \underline{y}^{n-l})$ as before where $p = (0, \ldots, 0)$, where the coordinate fields $\frac{d}{d\underline{x}^i}$ are just the action fields of $\mathcal{G}_\alpha$, and where the $\underline{y}^1, \ldots, \underline{y}^{n-l}$ are constant on the orbits of $\mathcal{G}_\alpha$. Note that $\frac{\partial \rho_i}{\partial \underline{x}_i} = 0$.

First we scale the coordinates

$$x^i = \underline{x}^i \log \delta \qquad\qquad y^i = \underline{y}^i \log \delta.$$

In the new coordinates, we still have $\frac{d\rho_j}{dx^i} = 0$, but also

$$\begin{aligned}
\frac{d\rho_j}{dy^i} &= \frac{1}{\log 2} \frac{df_j}{d\underline{y}^i} \frac{1}{f_j} \delta^{\log_2 f_j} \\
\frac{d^2\rho_j}{dy^k dy^i} &= \frac{1}{\log \delta} \frac{1}{\log 2} \frac{d^2 f_j}{d\underline{y}^k d\underline{y}^i} \frac{1}{f_j} \delta^{\log_2 f_j} - \frac{1}{\log \delta} \frac{1}{\log 2} \frac{df_j}{d\underline{y}^k} \frac{df_j}{d\underline{y}^i} \frac{1}{f_j^2} \delta^{\log_2 f_j} \\
&\quad + \left(\frac{1}{\log 2}\right)^2 \frac{df_j}{d\underline{y}^k} \frac{df_j}{d\underline{y}^i} \frac{1}{f_j^2} \delta^{\log_2 f_j}.
\end{aligned}$$

Therefore in these coordinates, the functions ρ_j'/ρ_j and ρ_j''/ρ_j are bounded as $\delta \to 0$. Since by the induction assumption the previous metric g_{j-1} has bounded curvature, so does the new metric.

□

3 Nonpolarized Collapse

Let $\mathcal{G}$ be an F-structure on the manifold M. We construct what is called a 'slice polarization.'

3.1 Pure structure

Let Σ_i be the union of orbits of $\mathcal{G}$ of dimension i. Let Σ_{ϵ_i} denote the set of points of Σ_i a distance of ϵ_i or greater from $\partial\Sigma_i$ (this is a "thickening" of Σ_i). If N is any submanifold let $\nu(N)$ denote the normal bundle. Let S_{ϵ_i, r_i} denote the set $\{v \in \nu(\Sigma_{\epsilon_i})\, s.t.\, \|v\| < r_i\}$, and let Σ_{ϵ_i, r_i} denote the image of S_{ϵ_i, r_i} under the exponential map. If r_i is chosen small enough, the exponential map is a diffeomorphism.

Lemma 3.1 *There is an invariant metric g and numbers ϵ_i, r_i so that*

(1) $\bigcup \Sigma_{\epsilon_i, r_i} = M$

(2) If $i < j$, then $\pi_i = \pi_i \circ \pi_j$ on $\Sigma_{\epsilon_i, r_i} \cap \Sigma_{\epsilon_j, r_j}$.

□

Now set $U_i = \Sigma_{\epsilon_i, r_i}$. If $q \in U_i$, then parallel translation from q to $\pi_i(q)$ along a geodesic induces an injection $\mathcal{G}_q \to \mathcal{G}_{\pi_i(q)}$.

Lemma 3.2 *There exists an inner product $\langle , \rangle_p$ on g_p, the Lie algebra of stalks $\mathcal{G}_p$, that is invariant under the action of $\mathcal{G}_p$ and under the projections π_i whenever $\pi_i(q)$ is defined.*

□

For $p \in S_{\epsilon_i, r_i}$ let K_p^i be the (not necessarily closed) subgroup of $\mathcal{G}_p$ whose lie algebra is the orthogonal complement of the isotropy group of p. Set $K_p^i = \pi_i^{-1}(K_{\pi_i(p)})$. It follows from the previous lemmas that the assignment $p \to K_p^i$ is invariant under the local action of $\mathcal{G}_p$.

We can now describe the collapsing procedure. Let f_i, ρ_i be as before. Fix q and let $U_{i_1}, \ldots, U_{i_j}$, $i_1 < \cdots < i_j$ be the U_i with $q \in U_i$. Let $Z_{i_1} \subseteq \cdots \subseteq Z_{i_j}$ denote the subspaces of $T_q M$ tangent to the orbits of $K_q^{i_1}, \ldots, K_q^{i_j}$. Let $W_{i_j} \subseteq \cdots \subseteq W_{i_1}$ denote the subspaces $W_{i_1} = \pi_{i_1}^{-1}(\mathcal{O}_{\pi_{i_1}(q)}), \ldots, W_{i_j} = \pi_{i_j}^{-1}(\mathcal{O}_{\pi_{i_j}(q)})$. Note that also $Z_{i_j} \subseteq W_{i_j}$.

Now let g be the invariant metric from Lemma 3.1. Set $g_0 = \log^2 \delta \cdot g$, and write a decomposition for g_0

$$g_0 \; = \; g_1' + h_1 + k_1,$$

corresponding to $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Put

$$g_1 \quad = \quad \begin{cases} \rho^2 g_1' + h_1 + \rho^{-2} k_1 & p \in U_1 \\ g_0 & \text{otherwise} \end{cases}$$

Proceed by induction, letting $g_{l-1} = g_l' + h_l + k_l$ be the decomposition according to Z_{i_l}, $Z_{i_l}^{\perp} \cap W_{i_l}$, $W_{i_l}^{\perp}$, and putting

$$g_l \quad = \quad \begin{cases} \rho^2 g_l' + h_l + \rho^{-2} k_l & p \in U_l \\ g_{l-1} & \text{otherwise} \end{cases}$$

First we claim that curvature is bounded as $\delta \to 0$. We establish a coordinate system. Let

$$m_i \; = \; \dim \Sigma_i \, - \, i \; = \; \dim \Sigma_i \, - \, \mathrm{rank}_{\mathbb{F}} \Sigma_i,$$

and let $s^1, \ldots, s^{m_{i_1}}$ be coordinates on Σ_{i_1} constant on the orbits. Extend these to U_{i_1} via π_{i_1}. Let $s^{m_{i_1}+1}, \ldots, s^{m_{i_2}}$, be coordinates on U_{i_2}, constant on the orbits. Extend these to $U_{i_1} \cap U_{i_2}$. Proceed in this way, finally getting coordinates $s^1, \ldots, s^{m_{i_j}}$ on $U_{i_1} \cap \cdots \cap U_{i_j}$. Now compliment these coordinates with additional coordinates $t^1, \ldots, t^{n-i_j-m_{i_j}}$ that are constant on the orbits of $K_q^{i_j}$ and so that $s^1, \ldots, s^{i_j}, t^1, \ldots, t^{n-i_j-m_{i_j}}$ is a complete system that is transverse to the orbits of $\mathbb{F}$. Finally let $x^1, \ldots x^{i_j}$ be coordinates so that $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_k}}$ are fields generated by the action of $K_q^{i_k}$.

Now we compute the curvature. First consider the change of metric $g_0 \mapsto g_1$. Relabel the coordinates

$$\begin{aligned} z^1 &= s^1 \\ &\vdots \\ z^{m_{i_1}} &= s^{m_{i_1}} \\ y^1 &= s^{m_{i_1}+1} \\ &\vdots \\ y^{m_{i_j}-m_{i_1}} &= s^{m_{i_j}} \\ y^{m_{i_j}-m_{i_1}+1} &= t^1 \\ &\vdots \\ y^{n-i_j-m_{i_1}} &= t^{n-i_j-m_{i_j}} \\ x^1 & \\ &\vdots \\ x^{i_j}. & \end{aligned}$$

The orthogonal decomposition of the tangent space given by $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$ roughly corresponds to the selection of the x, y, z coordinates. Working in the Σ_{i_1} stratum, $x^1, \ldots, x^{i_1}$ are coordinates on the rank i_1 orbits themselves; this roughly corresponds to Z_{i_1}. The subspace $Z_{i_1}^{\perp} \cap W_{i_1}$ is the subspace directly perpendicular to the stratum; this essentially parametrizes the orbits of $\mathbb{F}$ not in Σ_{i_1}, that is, captures the y coordinates, and also captures the remaining x^k. Finally $W_{i_1}^{\perp}$ parametrizes the orbits of Σ_{i_1}; in fact the coordinate functions $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_j}}$ project to zero in this space, or else the action of some of the other strata $\Sigma_{i_2}, \ldots, \Sigma_{i_j}$ would act on the Σ_1 stratum, which is impossible, and also the action fields are tangent to the orbits and $W_{i_1}^{\perp}$ is perpendicular to all obits. Since the y are coordinates on strata and the strata are perpendicular to $W_{i_1}^{\perp}$, we get that $\frac{d}{dy^k}$ is perpendicular to $W_{i_1}^{\perp}$ as well.

Thus we decompose the vectors

$$
\begin{aligned}
\frac{d}{dx} &= b_x^1 v_{x,1} + b_x^2 v_{x,2} \\
\frac{d}{dy} &= b_y^1 v_{y,1} + b_y^2 v_{y,2} \\
\frac{d}{dz} &= b_z^1 v_{z,1} + b_z^2 v_{z,2} + b_z^3 v_{z,3}
\end{aligned}
$$

according to the decomposition $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Multiplying the coordinate functions by $\log\delta$, we have again that $|\rho''_{i_1}/\rho_{i_1}|$ and $|\rho'_{i_1}/\rho_{i_1}|$ are bounded. We get the following matrix for g.

$$
(\log\delta)^2\, g = \begin{pmatrix} (b_x^1)^2 + (b_x^2)^2 & b_x^1 b_y^1 + b_x^2 b_y^2 & b_x^1 b_z^1 + b_x^2 b_z^2 \\ b_x^1 b_y^1 + b_x^2 b_y^2 & (b_y^1)^2 + (b_y^2)^2 & b_y^1 b_z^1 + b_y^2 b_z^2 \\ b_x^1 b_z^1 + b_x^2 b_z^2 & b_y^1 b_z^1 + b_y^2 b_z^2 & (b_z^1)^2 + (b_z^2)^2 + (b_z^3)^2 \end{pmatrix}
$$

therefore

$$
g_1 = \begin{pmatrix} \rho^2 (b_x^1)^2 + (b_x^2)^2 & \rho^2 b_x^1 b_y^1 + b_x^2 b_y^2 & \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 \\ \rho^2 b_x^1 b_y^1 + b_x^2 b_y^2 & \rho^2 (b_y^1)^2 + (b_y^2)^2 & \rho^2 b_y^1 b_z^1 + b_y^2 b_z^2 \\ \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 & \rho^2 b_y^1 b_z^1 + b_y^2 b_z^2 & \rho^2 (b_z^1)^2 + (b_z^2)^2 + \rho^{-2} (b_z^3)^2 \end{pmatrix}
$$

We make the change of coordinates $x \mapsto \rho_{i_1} x$, $z \mapsto \rho_{i_1}^{-1} z$. In the new coordinates the matrix reads

$$
g_1 = \begin{pmatrix} (b_x^1)^2 + \rho^{-2} (b_x^2)^2 & \rho b_x^1 b_y^1 + \rho^{-1} b_x^2 b_y^2 & \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 \\ \rho b_x^1 b_y^1 + \rho^{-1} b_x^2 b_y^2 & \rho^2 (b_y^1)^2 + (b_y^2)^2 & \rho^3 b_y^1 b_z^1 + \rho b_y^2 b_z^2 \\ \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 & \rho^3 b_y^1 b_z^1 + \rho b_y^2 b_z^2 & \rho^4 (b_z^1)^2 + \rho^2 (b_z^2)^2 + (b_z^3)^2 \end{pmatrix}.
$$

We must deal with the $\rho^{-1} b_x^2$ term somehow. As we choose δ differently, the b_x^2 (and the other b_K^i for $K = x, y, z$, $i = 1, 2, 3$) will be different. Let $b_{x,\delta}^2$ denote b_x^2 in he metric g_δ. Since the coordinate fields d/dx^k, $1 \le k \le i_1$, are inside of Z_{i_1} to first order, we get

$$
\begin{aligned}
\lim \frac{b_x^2(q)}{\rho} &= \lim_{\delta\to 0} \frac{b_{x,\delta}^2(q) - b_{x,0}^2(q)}{\rho - 0} \\
&= \lim_{\delta\to 0} \frac{1}{\log\delta} \frac{b_{x,\delta}^2(q) - b_{x,0}^2(q)}{\delta/\log\delta - 0} \\
&= 0
\end{aligned}
$$

Letting $\delta \to 0$, the limiting matrix is just

$$
g_1 = \begin{pmatrix} (b_x^1)^2 & 0 & 0 \\ 0 & (b_y^2)^2 & 0 \\ 0 & 0 & (b_z^3)^2 \end{pmatrix}.
$$

To continue, we now focus attention on U_{i_2}. We readjusts choice of coordinates, so now

$$\begin{aligned} z^1 &= s^1 \\ &\vdots \\ z^{m_{i_2}} &= s^{m_{i_2}} \\ y^1 &= s^{m_{i_2}+1} \\ &\vdots \\ y^{m_{i_j}-m_{i_2}} &= s^{m_{i_j}} \\ y^{m_{i_j}-m_{i_2}+1} &= t^1 \\ &\vdots \\ y^{n-i_j-m_{i_2}} &= t^{n-i_j-m_{i_j}} \\ x^1 & \\ &\vdots \\ x^{i_j}. & \end{aligned}$$

One considers the splitting of the tangent space via $Z_{i_2}, Z_{i_2}^{\perp} \cap W_{i_2}, W_{i_2}^{\perp}$, and repeats the computation of the curvature matrix as above.

To see collapse, the idea is that the orbits are almost totally geodesic as the collapsing proceeds. To be specific, let $q \in M$ and let $\mathcal{O}_q$ be its orbit. Choose r so that the exponential map on vectors perpendicular to the orbits is a diffeomorphism on vectors of length $< r$. There is a number c so that $\mathrm{dist}(q, \partial T_{r/2}(\mathcal{O}_q)) > c$. However there is a closed loop that is noncontractible in $T_{r/2}(\mathcal{O}_q)$ and has length $< c'\delta$. For $\delta < c/c'$ this implies there is a noncontractible geodesic in $T_{r/2}(\mathcal{O}_q)$ of length $< c'\delta$; hence the injectivity radius converges to 0 at q.

3.2 Nonpure collapse

If the structure is not pure, then we work on a regular atlas $U_1, \ldots, U_A$. Over U_α we have a pure substructure G_α, and we can carry out the procedure above. If we order the atlases so $\mathcal{G}_{1,p} \subset \cdots \subset \mathcal{G}_{A,p}$, then the orbit stratification near p for higher U_α refines that for lower U_α. We must also modify the cutoff functions ρ_i^α to be equal to 1 in some neighborhood of ∂U_α; this way the charts in the atlas are pushed away from each other as well as the strata inside each chart.