

MATH 371 REVIEW 1

Some Notation and Basic Concepts

Linear Algebra

- fields.
- Examples of fields:
 - the field of rational, real and complex numbers: \mathbb{Q} , \mathbb{R} and \mathbb{C}
 - $\mathbb{F}_p :=$ the finite field with p elements, where p is a prime number
 - the field of rational functions, $\mathbb{Q}(x)$, $\mathbb{R}(x)$, $\mathbb{C}(x)$, consisting of quotients of polynomials with coefficients in \mathbb{Q} , \mathbb{R} , \mathbb{C} respectively.
- vector spaces, subspaces.
- dimension, basis, linear independence.
- linear transformations (also called “homomorphisms between vector spaces”)
- matrix representations of a linear transformation. (So, after choosing a basis of the source and a basis of a target vector space, a linear transformation is represented by a matrix.) Effect of change of bases on the matrix representation.
- direct sum of vector spaces
- eigenvalues, eigenvectors, and eigenspaces (of a linear transformation), characteristic polynomials.
- diagonalization (of a linear transformation)

Basic Group Theory

- groups, subgroups
- Examples of groups
 - cyclic groups: $\mathbb{Z} :=$ integers, $\mathbb{Z}/n\mathbb{Z} =$ a cyclic group with n elements.
 - the dihedral group D_{2n} with $2n$ elements; it is the group of all symmetries of a regular n -gon.
 - the quaternion group Q_8 with 8 elements.
 - the “general linear groups” $\mathrm{GL}_n(\mathbb{Q})$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{F}_p)$. These are the group of invertible $n \times n$ matrices with coefficients in \mathbb{Q} , \mathbb{R} , \mathbb{C} , and \mathbb{F}_p respectively. Notice that $\mathrm{GL}_1(\mathbb{Q}) = \mathbb{Q}^\times$, the group of non-zero rational numbers under multiplication; similarly $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, $\mathrm{GL}_1(\mathbb{F}_p) = \mathbb{F}_p^\times$.

- the “special linear groups” $SL_n(\mathbb{Q})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$ and $SL_n(\mathbb{F}_p)$; each is the subgroup of the respective general linear group consisting of all elements whose determinant is equal to 1.
 - the “projective linear groups”: $PGL_n(\mathbb{Q}) := GL_n(\mathbb{Q})/\mathbb{Q}^\times \cdot I_n$, the other groups $PGL_n(\mathbb{R})$, $PGL_n(\mathbb{C})$, and $PGL_n(\mathbb{F}_p)$ are defined similarly.
 - the symmetric group S_n with $n!$ elements, consisting of all permutations of the set $\{1, 2, \dots, n\}$.
 - the alternating group A_n , equal to the subgroup of index 2 in A_n consisting of all *even* permutations in S_n .
- cosets
 - homomorphisms of groups
 - $\text{Ker}(h) :=$ the kernel of a homomorphism $h : G \rightarrow G'$ between groups, $\text{Image}(h) :=$ the image of a homomorphism $h : G \rightarrow G'$ between groups.
 - normal subgroups, quotient groups
 - the product of (a finite number of) groups
 - conjugacy class (of an element in a group)
 - $Z_G(x) :=$ the centralizer of an element x in a group G , $Z_G(H) :=$ the centralizer of a subgroup H of a group G .
 - $N_G(x) :=$ the normalizer of an element x in a group G , $N_G(H) :=$ the normalizer of a subgroup H in a group G .

Review Problems

1. Let V be the \mathbb{R} -vector space consisting of all polynomials of degree at most 3 with coefficients in \mathbb{R} . Let $D : V \rightarrow V$ be the linear transformation which sends every polynomial $f(x)$ in V to the derivative $\frac{df}{dx}$. Let $\partial : V \rightarrow V$ be the linear transformation which sends every polynomial $f(x)$ in V to $x \frac{df}{dx}$.
 - (i) Show that $\{1, x, x^2, x^3\}$ form a basis of V .
 - (ii) Find the matrix representation of D and ∂ with respect to the basis in (i) above.
 - (iii) Let n be a positive integer. Find a basis of $\text{Ker}(D^n)$ and a basis of $\text{Image}(D^n)$
 - (iv) Find $\text{Ker}(\partial^n)$ and $\text{Image}(\partial^n)$ for a positive integer n .
 - (v) Find the eigenvalues of D . Is D diagonalizable?

- (vi) Find the eigenvalues of ∂ . Is ∂ diagonalizable?
2. Prove that \mathbb{F}_7^\times is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, i.e. it is cyclic of order 6.
3. Let G be a group and let a, b be two elements in G . Prove that ab and ba have the same order. (In other words, if ab has finite order, so ba , and the two orders are equal.)
4. Let $G = \text{GL}_2(\mathbb{R})$, the group of invertible 2×2 matrices with coefficients in \mathbb{R} . Let

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R}^\times \right\}$$

be the subgroup of diagonal matrices in G .

- (i) Find the center $Z(G)$ of G .
- (ii) Determine the centralizer $Z_G(H)$ of H in G .