MATH 4250 PROBLEM SET 1, SPRING 2024

Part 1. From Strauss, *Partial Differential Equations*, Chapter 1.

- Exercise 1.1, #9
- Exercise 1.2, #2, #9

Warning: Exercises in Strauss are at the end of each section. Unfortunately these end-of-section exercises are not clearly labeled with section numbers. So **please be careful in identifying these exercises.** For instance exercise 1.1 are on pages 5–6, exercises 1.2 are on pages 9–10. This warning will not be repeated in other problem sets.

Part 2.

1. Find the general solution of $u_{xy} = x^2 y$ for the function $u(x, y)$ on $\mathbb{R}^2$.

2. Find the general solution for $yu_{xy} + 2u_x = x$ for the function $u(x, y)$ on $\mathbb{R}^2$. (Hint: first integrate with respect to x.)

3. Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded (connected) region with smooth boundary $\mathcal{B}$. If $u(x, y)$ is a "smooth" function, write $\Delta u = u_{xx} + u_{yy}$ (we call Δ the *Laplace operator*). Some people write $\Delta u = \nabla^2 u$. Here we have used the standard notion for partial derivatives. For instance $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, and in the next problem $u_t = \dfrac{\partial u}{\partial t}$.

> SUGGESTION: First do this problem for a function of *one* variable, $u(x)$, so $\Delta u = u''$ and, say, $\mathcal{D}$ is the interval $\{0 < x < 1\}$.

a) Show that $u\Delta u = \nabla \cdot (u\nabla u) - |\nabla u|^2$.

b) If $u(x, y) = 0$ on $\mathcal{B}$. Show that

$$\iint_{\mathcal{D}} u\Delta u \, dx\, dy = -\iint_{\mathcal{D}} |\nabla u|^2 \, dx\, dy.$$

[Hint: You may want to use the divergence theorem the theorem on the plane, which is equivalent to Green's theorem.]

c) If $\Delta u = 0$ in $\mathcal{D}$ and $u = 0$ on the boundary $\mathcal{B}$, show that $u(x, y) = 0$ throughout $\mathcal{D}$.

4. (extra credit) The temperature $u(x, t)$ of a certain thin rod, $0 \le x \le L$ satisfies the *heat equation*

$$u_t = u_{xx} \tag{1}$$

Assume the initial temperature $u(x, 0) = 0$ and that both ends of the rod are kept at a temperature of 0, so $u(0, t) = u(L, t) = 0$ for all $t \ge 0$. What do you anticipate the temperature in the rod will be at any later time t?

I hope you suspect that $u(x,t) = 0$ for all $t \geq 0$. Use the following to prove this. Let

$$H(t) = \int_0^L u^2(x,t)\,dx.$$

a) Show that since the temperature on the ends of the rod is always zero, then $dH/dt \leq 0$ (an integration by parts will be needed). Thus, for any $t \geq 0$ we know that $H(t) \leq H(0)$.

b) Since the initial temperature is zero, what is $H(0)$? Why does this imply that $H(t) = 0$ for all $t \geq 0$? Why does this imply that $u(x,t) = 0$ for all points on the rod and all $t \geq 0$?

Uniqueness Say that the functionns $u(x,t)$ and $v(x,t)$ both satisfy the heat equation (1) and have the identical initial values and boundary values:

$$u(x,0) = v(x,0) \text{ for } 0 \leq x \leq L, \qquad u(x,t) = v(x,t) \text{ for } x = 0 \text{ and } x = L,\ t \geq 0.$$

Show that $u(x,t) = v(x,t)$ for all $0 \leq x \leq L$, $t \geq 0$.

5. (extra credit) [Generalization of Problem B to more space dimensions]. Say a function $u(x,y,t)$ satisfies the heat equation in a bounded region $\Omega \in \mathbb{R}^2$:

$$u_t = u_{xx} + u_{yy} \tag{2}$$

and that $u(x,y,t) = 0$ for all points (x,y) on the boundary, $\mathcal{B}$ of Ω. Similar to Problem 17, define

$$H(t) = \iint_\Omega u^2(x,y,t)\,dx\,dy.$$

a) Show that $dH/dt \leq 0$. [Suggestion: See Problem 16.]

b) If in addition you know that the initial temperature is zero, $u(x,y,0) = 0$ for all points $(x,y) \in \Omega$, show that $u(x,y,t) = 0$ for all $(x,y) \in \Omega$ and all $t \geq 0$.

c) [Uniqueness] Say that the functionns $u(x,y,t)$ and $v(x,y,t)$ both satisfy the heat equation (2) and have the identical initial values and boundary values:

$$u(x,y,0) = v(x,y,0) \text{ for } (x,y) \in \Omega, \qquad u(x,y,t) = v(x,y,t) \text{ for } (x,y) \text{ on } \mathcal{B}, \text{ and } t \geq 0.$$

Show that $u(x,y,t) = v(x,y,t)$ for all $(x,y) \in \Omega\ \ t \geq 0$.