Math 4250 Problem Set 6, Spring 2024

Part 1. From Strauss, *Partial Differential Equations*.

- Exercise 4.3, #6, page 100
- Exercise 5.1, #4, page 111
- Exercise 5.2, #15, page 118
- Exercise 5.3, #9, page 123
- Exercise 5.4, #15, page 135

Part 2.

1. (a) Let $f \in C(\mathbb{R}/\mathbb{Z})$ be a continuous periodic function. Show that there exists a positive constant M_0 such that $|c_n(f)| \leq M_0$ for all $n \in \mathbb{Z}$.

(b) Let $f \in C^1(\mathbb{R}/\mathbb{Z})$ be a continuously differentiable periodic function. Show that there exists a positive constant M_1 such that $|c_n(f)| \leq \frac{M_1}{|n|}$ for all $n \in \mathbb{Z}$ with $n \neq 0$. (Hint: integrate by parts)

(c) (extra credit) Let $f \in C^k(\mathbb{R}/\mathbb{Z})$ be a k-times continuously differentiable function. Show that there exists a positive constant M_k such that $|c_n(f)| \leq \frac{M_k}{|n|^k}$ for all $n \in \mathbb{Z}$ with $n \neq 0$.

Moral: The smoother a function is, the faster its Fourier coefficients decay.

2. The Haar functions $\left(e_n^k\right)_{n\in\mathbb{N},\,1\leq k\leq 2^n}$ are function on $[0,1]$ defined as follows: $e_0^0(x) = 1$ for all $x \in [0,1]$, and

$$e_n^k(x) = \begin{cases} 2^{n/2} & \text{if } \frac{k-1}{2^n} \leq x < \frac{2k-1}{2^{n+1}}, \\ -2^{n/2} & \text{if } \frac{2k-1}{2^{n+1}} \leq x < \frac{k}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the Haar functions form a set of orthonormal functions in $L^2([0,1])$.

(b) (extra credit) Show that if f is a continuous function on $[0,1]$ orthogonal to every Haar function, then $f = 0$, the 0-function.

3. (extra credit) Let $f \in L^2(\mathbb{R}/\mathbb{Z})$, a square integrable periodic function. For each positive integer $m \geq 1$, let f_m be the L^2-function

$$f_m(x) := \frac{1}{m}\sum_{k=0}^{m-1} f(x + \tfrac{k}{m})$$

(a) Find a simple expression of the Fourier coefficients $c_n(f_m)$ of f_m in terms of the Fourier coefficients of f.

(b) Show that $\lim_{m\to\infty} f_m = c_0(f) = \int_0^1 f(x)dx$ in $L^2(\mathbb{R}/\mathbb{Z})$, i.e. $||f_m - c_0(f)|| \to 0$ as $m \to \infty$.

[Hint: Use the discrete Plancherel identity, also known as the Parseval identity.]