

MATH 602 ASSIGNMENT 1, FALL 2006

1. Do Problem 1 of Shatz-Gallier.

2. Let p be a prime number, and let $n \geq 3$ be a positive integer. Define the modular group Mod_{p^n} with p^n elements to be the semi-direct product $(\mathbb{Z}/p^{n-1}\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z})$, where the generator $\bar{1} \in \mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{Z}/p^{n-1}\mathbb{Z}$ by $a \mapsto (p^{n-2} + 1)a$ for all $a \in \mathbb{Z}/p^{n-1}\mathbb{Z}$. Show that $\text{Mod}_8 \cong D_8$, the dihedral group with 8 elements.

3. Do Problem 13 of Shatz-Gallier.

Definition Suppose a group G operates transitively on the left of a set X .

(i) We say that the action (G, X) is *imprimitive* if there exists a non-trivial partition Q of X which is stable under G . Here “non-trivial” means that

$$Q \neq \{X\} \quad \text{and} \quad Q \neq \{\{x\} : x \in X\}.$$

Such a partition Q will be called a *system of imprimitivity* for (G, X) .

(ii) We say that the action (G, X) is *primitive* if it is not imprimitive.

4. (a) Suppose that (G, X, Q) is a system of imprimitivity for a transitive left action (G, X) , $Y \in Q$, $y \in Y$. Let $H = \text{Stab}_G(Y)$, $K = \text{Stab}_G(y)$. Prove the following statements.

- $K < H < G$ and H operates transitively on Y .
- $|X| = |Y| \cdot |Q|$, $|Q| = [G : H]$, $|Y| = [H : K]$. (These statements hold even if X is infinite.)

(b) Suppose that (G, X) is a transitive left action, $y \in X$, $K := \text{Stab}_G(y)$, and H is a subgroup of G such that $K < H < G$. Let $Y := H \cdot y \subset X$, and let $Q := \{g \cdot Y \mid g \in G\}$. Show that (G, X, Q) is a system of imprimitivity.

(c) Suppose that (G, X) is a transitive left action, $x \in X$. Show that (G, X) is primitive if and only if G_x is a maximal subgroup of G . (Note that a maximal subgroup of G is by definition a maximal proper subgroup of G .)

5. Give an example of a group G which does not have any maximal subgroup.

6. Let p be a prime number. For $n = 2, 3$, find a p -Sylow subgroup P of $\text{GL}_n(\mathbb{F}_p)$. Also, determine the cardinality of $\text{GL}_n(\mathbb{F}_p)$ and the normalizer of P in $\text{GL}_n(\mathbb{F}_p)$.

7. Let (G, X) be a transitive left action, $x \in X$, $H := \text{Stab}_G(x)$. Let $K \leq H$ be a subgroup of H . Let $\text{Fix}(K)$ be the fixed point subset of K in X . Let ${}^H K$ be the family of H -conjugates of K . Denote by ${}^G K \cap H$ the family of G -conjugates of K which lie in H . Show that $N_G(K)$ operates transitively on $\text{Fix}(K)$ if and only if ${}^H K = {}^G K \cap H$.

8. (a) Suppose that (G, X) is a transitive left G action, $x \in X$, and H is a subgroup of G . Let $K = \text{Stab}_G(x)$. Show that H operates transitively on X if and only if $G = H \cdot K$. Here

$$H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$$

(b) (Frattini argument) Let H be a subgroup of K , and let X be a subset of H . Denote by ${}^K X$ (resp. ${}^H X$) the set of all K -conjugates of X (resp. the set of all H -conjugates of X). Prove that $K = H \cdot N_K(X)$ if and only if ${}^K X = {}^H X$.

(c) (Frattini argument) Let H be a normal subgroup of G , p be a prime number, and let P be a p -Sylow subgroup of H . Prove that $G = H \cdot N_G(P)$.

(d) Let G be a finite group and let $\Phi(G)$ be the Frattini subgroup of G . Show that every Sylow subgroup of $\Phi(G)$ is normal in $\Phi(G)$. Conclude that $\Phi(G)$ is the direct product of its Sylow subgroups. Moreover $\Phi(G)$ has property N. (See §1.3 of Chap. 1 of Shatz-Gallier.)