# Math 602 Assignment 10, Fall 2006

1. Compute the character table of the quaternion group with 8 elements.

2. Let $N$ be the subgroup of $\mathrm{GL}_3(\mathbb{F}_3)$ consisting of all upper-triangular unipotent $3\times3$ matrices with entries in $\mathbb{F}_3$. Determine the character table of $N$.

3. Determine the character table of the quaternion group $Q_{16}$ with 16 elements defined in Assignment 2.

4. Let $G$ be a finite group whose cardinality is an odd number. Prove that $\chi^{(i)} \neq \overline{\chi^{(i)}}$ for any nontrivial complex irreducible character $\chi^{(i)}$. (Hint: Use a suitable orthogonality relation to show that if $\chi^{(i)} = \overline{\chi^{(i)}}$, then $\frac{\chi^{(i)}(1)}{2}$ would be an algebraic integer.)

5. Let $G$ be a finite group and let $\chi^{(i)}$ be the character of an irreducible linear representation of $G$ on a finite dimensional vector space $V$ over $\mathbb{C}$. Let $n = \mathrm{Card}(G)$.

(i) Show that $\chi(\sigma)$ is an algebraic integer in the cyclotomic field $\mathbb{Q}(\mu_n)$ for every $\sigma \in G$.

(ii) Let $\sigma$ be an element of $G$ such that $\chi(\sigma) \neq 0$. Show that

$$\mathrm{Tr}_{\mathbb{Q}(\mu_n)/\mathbb{Q}}(\chi(\sigma)\cdot\bar{\chi}(\sigma)) \geq [\mathbb{Q}(\mu_n):\mathbb{Q}]\,.$$

(Hint: Use the fact that the arithmetic means of a finite number of elements in $\mathbb{R}_{\geq 0}$ is bigger than or equal to their geometric means.)

(iii) Assume that $\dim_{\mathbb{C}}(V) > 1$. Prove that there exists an element $\sigma \in G$ such that $\chi(\sigma) = 0$.

(iv) Let $\tau$ be an automorphism of the cyclotomic field $\mathbb{Q}(\mu_n)$. Show that there exists an integer $a$ such that $\tau(\chi(\sigma)) = \chi(\sigma^a)$ for every $\sigma \in G$.

6. (extra credit) Determine the character table of the finite group $\mathrm{SL}_2(\mathbb{F}_5)$.