Math 602 Assignment 2, Fall 2006

Definition. (i) Let $n \geq 3$ be a positive integer. Recall that the dihedral group D_{2^n} is the semi-direct product $(\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_\rho (\mathbb{Z}/2\mathbb{Z})$ for the action $\rho : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/2^{n-1}\mathbb{Z})$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/2^{n-1}\mathbb{Z}$ as $a \mapsto -a \ \forall a \in \mathbb{Z}/2^{n-1}\mathbb{Z}$.

(ii) Let $n \geq 4$ be a positive integer. Let SD_{2^n} be the semi-direct product $(\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/2\mathbb{Z})$ for the action $\tau : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/2^{n-1}\mathbb{Z})$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/2^{n-1}\mathbb{Z}$ as $a \mapsto (2^{n-2}-1)a \ \forall a \in \mathbb{Z}/2^{n-1}\mathbb{Z}$.

(iii) Let $n \geq 3$ be a positive integer. Let G be the semi-direct product $(\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_\mu (\mathbb{Z}/4\mathbb{Z})$ for the action $\mu : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/2^{n-1}\mathbb{Z})$, where the generator $\bar{1} = 1 + 4\mathbb{Z}$ of $\mathbb{Z}/4\mathbb{Z}$ operates on $\mathbb{Z}/2^{n-1}\mathbb{Z}$ as $a \mapsto -a \ \forall a \in \mathbb{Z}/2^{n-1}\mathbb{Z}$. Notice that $\mathrm{Z}(G)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, generated by the elements $(2^{n-2} + 2^{n-1}\mathbb{Z}, 4\mathbb{Z})$, $(2^{n-1}\mathbb{Z}, 2 + 4\mathbb{Z})$. Define the *quaternion group* Q_{2^n} with 2^n elements to be the quotient of G by the subgroup of $\mathrm{Z}(G)$ generated by the element $(2 + 2^{n-1}\mathbb{Z}, 2 + 4\mathbb{Z})$ of order two.

1. For $G = \mathrm{Mod}_{p^n}$ determine the Frattini subgroup $\Phi(G)$, the commutator subgroup $[G, G]$, the p-rank of G (i.e. the largest integer m such that G contains a subgroup isomorphic to p^m), the ascending central series of G and the descending central series of G.

2. For $G = \mathrm{D}_{2^n}$, SD_{2^n} or Q_{2^n}, determine the Frattini subgroup $\Phi(G)$, the commutator subgroup $[G, G]$, the 2-rank of G, the ascending central series of G and the descending central series of G.

3. Suppose that G is a non-commutative (finite) p-group for a prime number p such that G has a cyclic normal subgroup of index p. Prove that G is isomorphic to a Mod_{p^n} if p is odd, and G is isomorphic to a Mod_{2^n}, D_{2^n}, SD_{2^n} or Q_{2^n} if $p = 2$.

4. Let G be a finite p-group, where p is a prime number. Show that there is a characteristic subgroup H of G such that $\Phi(H) \leq \mathrm{Z}(H) = \mathrm{Z}_G(H) \geq [G, H]$. Here $[G, H]$ is the subgroup of G generated by commutators of the form $x^{-1}y^{-1}xy$ with $x \in G, y \in H$. A subgroup of G with the above properties is called a *critical subgroup* of G. (Hint: Consider the partially order set $\mathfrak{S}$ of all characteristic subgroups H of G such that $\Phi(H) \leq \mathrm{Z}(H) \geq [G, H]$, and show that every maximal element in $\mathfrak{S}$ is a critical subgroup of G.)

5. Find a critical subgroup for each of the following groups: Mod_{p^n}, D_{2^n}, SD_{2^n}, Q_{2^n}.

6. Let $G = S_3 = \mathrm{D}_3$, and let N be the normal subgroup with 3 elements in G. Denote by e_N the element

$$e_N := \frac{1}{3} \sum_{x \in N} x \in \mathbb{Q}[G]$$

in the rational group algebra $\mathbb{Q}[G]$ of G.

(i) Show that $e_N \in \mathrm{Z}(\mathbb{Q}[G])$, and $e_N^2 = e_N$. Consequently the ideals $e_N\mathbb{Q}[G] = e_N\mathbb{Q}[G]e_N = \mathbb{Q}[G]e_N$ and $(1-e_N)\mathbb{Q}[G] = (1-e_N)\mathbb{Q}[G](1-e_N) = \mathbb{Q}[G](1-e_N)$ have natural structure as rings with unit, and we have a natural isomorphism

$$\mathbb{Q}[G] = e_N\mathbb{Q}[G]e_N \times (1 - e_N)\mathbb{Q}[G](1 - e_N)$$

of $\mathbb{Q}$-algebras.

(ii) Prove that the map $x \to e_N x$ defines a surjective ring homomorphism π from $\mathbb{Q}[G]$ to $\mathbb{Q}[G/N] \cong \mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$ whose kernel is equal to $(1-e_N)\mathbb{Q}[G](1-e_N)$. Show that $\mathbb{Q}[G/N]$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}$ as $\mathbb{Q}$-algebras.

(iii) Show that $\mathbb{Q}[G]/e_N\mathbb{Q}[G]e_N \cong (1-e_N)\mathbb{Q}[G](1-e_N)$ is a four-dimensional simple $\mathbb{Q}$-algebra, i.e. it has no non-trivial two-sided ideals.

(iv) Is $\mathbb{Q}[G]/e_N\mathbb{Q}[G]e_N$ isomorphic to $\mathrm{M}_2(\mathbb{Q})$? (Either establish an isomorphism or show that no such isomorphism exists.)

7. Let $p \geq 5$ be prime number. Let $G = \mathrm{D}_{2p}$, the dihedral group with $2p$ element. Let N be the normal cyclic subgroup of order p in G. Let $e_N := \frac{1}{p}\sum_{x\in N} x \in \mathbb{Q}[G]$.

(i) Show that $e_N \in \mathbb{Q}[G]$ and $e_N^2 = 1$.

(ii) Let $\pi : \mathbb{Q}[G] \to \mathbb{Q}[G/N] \cong \mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$ be the surjective ring homomorphism induced by the canonical surjection $G \twoheadrightarrow G/N$. Show that $(1-e_N)$ generates $\mathrm{Ker}(\pi)$.

(iii) The ideal $(1-e_N)\mathbb{Q}[G] = (1-e_N)\mathbb{Q}[G](1-e_N) = \mathbb{Q}[G](1-e_N)$ of $\mathbb{Q}[G]$ has a natural structure as a $\mathbb{Q}$-algebra; denote it by A. Show that A is isomorphic to $\mathbb{Q}[G]/e_N\mathbb{Q}[G]$ as a $\mathbb{Q}$-algebra. Prove that the center of $\mathbb{Q}[G]/e_N\mathbb{Q}[G]$ is equal to the field $\mathbb{Q}[N]/e_N\mathbb{Q}[N] =: F$.

(iv) Is $\mathbb{Q}[G]/e_N\mathbb{Q}[G]$ isomorphic to $\mathrm{M}_2(F)$? (Either establish an isomorphism or show that no such isomorphism exists.)

8. Let $Q = \mathrm{Q}_8$ be the quaternion group with 8 elements. The center $Z = \mathrm{Z}(Q)$ of Q is generated by the unique element σ of order 2 in Q. Let $e_Z := \frac{1+\sigma}{2} \in \mathbb{Q}[Q]$. Denote by A the $\mathbb{Q}$-algebra $\mathbb{Q}[Q]/e_Z\mathbb{Q}[Q]$.

(i) Show that the center of A is $\mathbb{Q}$, i.e. $\dim_{\mathbb{Q}}(\mathrm{Z}(A)) = 1$, and $\dim_{\mathbb{Q}}(A) = 4$.

(ii) Is A isomorphic to $\mathrm{M}_2(\mathbb{Q})$? (Either establish an isomorphism or show that no such isomorphism exists.)