MATH 602 ASSIGNMENT 3, FALL 2006

1. Let k be a field. Denote by R the quotient ring $R = k[x,y]/(x^2+y^2, xy)$. Let V be the k-vector space underlying R. Denote by T_x (resp. T_y, resp. T_{x+y}) the element of $\mathrm{End}_k(V)$ given by $v \mapsto x \cdot v$ (resp. $v \mapsto y \cdot v$, resp. $v \mapsto (x+y)\cdot v$) for $v \in V$. Find a k-basis B of V the matrix representation of T_x, T_y and T_{x+y} is the Jordan canonical form of these linear operators.

2. Let G_1, G_2 be finite groups, and let k be a commutative ring. Show that the tensor product $k[G_1] \otimes_k k[G_2]$ is isomorphic to $k[G_1 \times G_2]$.

3. Let $h : G \to H$ be a homomorphism between finite groups. Let k be a field. Let I be the kernel of the ring homomorphism $\rho : k[G] \to k[H]$ induced by h.

(i) Suppose that $|G| \in k^\times$. Show that there exists an element $u \in I \cap Z(k[G])$ such that $I = uk[G]$.

(ii) Suppose that $|G| \cdot 1 = 0$ in k. Is the statement in (i) still true? (Give a proof or produce a counter-example.)

4. Let R be a commutative ring, and let M, N be R-modules.

(i) Assume that M is a finitely generated projective R-module. Prove that the canonical map

$$\mathrm{Hom}_R(M,R) \otimes_R M \to \mathrm{Hom}_R(M,N)$$

is an isomorphism of R-modules. (See pp. 69–71 of Shatz-Gallier for the definition of projective modules.)

(ii) Show by counter-examples that the the statement in (i) does not hold if the either the finite generation hypothesis or the projectivity hypothesis is dropped. Can you produce examples such that the canonical map in (i) is neither injective nor surjective?

5. Let R be a ring. Let M be a finitely generated projective left R-module. Denote by $M^\vee$ the R-linear dual of M, i.e. $M^\vee := \mathrm{Hom}_R(M,R)$. Notice that $M^\vee$ has a natural structure

(i) Show that the map

$$M^\vee \times M \to \mathrm{End}_R(M) \qquad (v^\vee, v) \ \mapsto \ (m \mapsto v^\vee(m)v \ \ \forall m \in M)$$

induces an isomorphism

$$M^\vee \otimes_R M \xrightarrow{\sim} \mathrm{End}_R(M)$$

of abelian groups.

(ii) Assume that R is a commutative ring. Show that the map

$$\mathrm{ev} : M^\vee \otimes_R M \to R, \qquad \sum x_i^\vee \otimes y_i \mapsto \sum_i x_i^\vee(y_i)$$

where $x_i^\vee x \in M^\vee$ and $y_i \in M$ for all i, is a well-defined map of R-modules. Composing the map ev with the inverse of the canonical isomorphism in (i) above, one obtains an R-linear map

$$\mathrm{Tr} : \mathrm{End}_R(M) \to R\,.$$

(iii) Assume that M is a free R-module of finite rank and R is commutative. Show that the map Tr defined in (ii) coincides with the standard definition in terms of matrix representations.

(iv) Assume that R is commutative. Prove that that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in \mathrm{End}_R(M)$.

(v) Assume that R is commutative. Let M_1, M_2 be finitely generated projective R modules. To a pair (T_1, T_2) with $T_1 \in \mathrm{End}_R(M_1)$, $T_2 \in \mathrm{End}_R(M_2)$, we have the attached R-linear operators $T_1 \oplus T_2 \in \mathrm{End}_R(M_1 \oplus M_2)$ and $T_1 \otimes T_2 \in \mathrm{End}_R(M_1 \otimes M_2)$. Express $\mathrm{Tr}(T_1 \oplus T_2)$ and $\mathrm{Tr}(T_1 \otimes T_2)$ in terms of $\mathrm{Tr}(T_1)$ and $\mathrm{Tr}(T_2)$.

(vi) Assume that R is a non-commutative division ring. Is the map ev in (ii) well-defined? Can you find an alternative definition of Tr which is also valid in the present situation?

6. Let k be a commutative ring, n be a positive integer. Let $R = \mathrm{M}_n(k)$, M be the set of all row vectors of length n with entries in k. Let N be the set of all column vectors of length n with entries in k. Note that M has a natural structure as a right R-module under matrix multiplication, while N has a natural structure as a left R-module under matrix multiplication. Show that the map $M \times N \to k$ given by matrix multiplication induces an isomorphism $M \otimes_R N \xrightarrow{\sim} k$.