Math 602 Assignment 4, Fall 2006

1. Let R be a ring, M, N be left R-modules, and let $\alpha : M \to N$ be an R-linear map. Assume that the map $\mathrm{Hom}_R(N,T) \to \mathrm{Hom}_R(M,T)$ induced by α is surjective for every left R-module T. Prove that α is injective, and there exists an R-submodule P of N such that $\alpha(M)+P = N$ and $\alpha(M) \cap P = (0)$.

2. (i) Let R, S be rings, M be a right R-module, N be an (R,S)-bimodule, and let P be a right S-module. Show that

$$\mathrm{Hom}_S(M \otimes_R N, P) = \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))\,,$$

i.e. exhibit a canonical isomorphism between both sides of the above equality.

(ii) Let T be a ring. In (i) above, assume that M is a (T,R)-bimodule, Prove that the canonical isomorphism in (i) is an isomorphism of right T-modules.

(iii) Formulate and prove an analog of (i) in which the first term (i.e. M) in the tensor product is a bimodule and the second term (i.e. N) is a left module.

(iv) Let $h : R \to S$ be a homomorphism of rings. Let L be a left R-module, and let Q be a left S-module. Show that there is a canonical bijection

$$\mathrm{Hom}_S(S \otimes_R L, Q) = \mathrm{Hom}_R(L, Q)$$

(On the left hand side, S is regarded as a left R module via the ring homomorphism h. On the right hand side, Q is regarded as a left R-module via h.)

3. Let k be a commutative ring. Let $h : k \to R$ be a ring homomorphism such that $h(k) \subseteq \mathrm{Z}(R)$.

(i) Show that $R \otimes \mathrm{M}_n(k) \cong \mathrm{M}_n(R)$ for every $n \geq 1$.

(ii) Show that $\mathrm{M}_m(k) \otimes_k \mathrm{M}_n(k) \cong \mathrm{M}_{mn}(k)$ for every $m, n \geq 1$.

4. Find a homomorphism $A \to B$ between commutative rings and two A-modules M, N such that the canonical map

$$B \otimes_A \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$$

is neither injective nor surjective. Can you find such an example in which M is a finitely generated projective A-module?

5. Let n be a positive integer. Let k be a commutative ring. Let $\underline{x}_n = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are "variables". Let $k[\underline{x} = k[x_1, \ldots, x_n]$ be the polynomial ring in $x_1, \ldots, x_n$. Similarly let $k[\underline{x}, \underline{x}']$ be the polynomial ring in the variables $x_1, \ldots, x_n, x_1', \ldots, x_n'$. Let $\Delta : k[\underline{x}] \to k[\underline{x}, \underline{x}']$ be the map defined by $\Delta(f(x_1, \ldots, x_n)) = f(x_1 + x_1', \ldots, x_n + x_n')$. We identify $k[\underline{x}, \underline{x}']$ with $k[\underline{x}] \otimes_k k[\underline{x}]$, so that x_i corresponds to $x_i \otimes 1$ and x_i' corresponds to $1 \otimes x_i$. Let σ be the involution on $k[\underline{x}] \otimes_k k[\underline{x}]$ interchanging the two factors.

(i) Show that Δ is co-commutative, i.e. $\sigma \circ \Delta = \Delta$.

(ii) Show that Δ is co-associative, i.e.

$$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$$

as maps from $k[\underline{x}]$ to the triple tensor product $k[\underline{x}] \otimes_k k[\underline{x}] \otimes_k k[\underline{x}]$.

(iii) Formulate a notion of "co-unity" of co-algebras, and show that $k[\underline{x}]$ has a co-unity.

(iv) Properties (i)—(iii) above can be summarized by saying that the map Δ give $k[\underline{x}]$ a structure as a co-algebra with co-unity which is co-commutative and co-associative. Being a polynomial algebra itself, the k-algebra $k[\underline{x}, \underline{y}]$ has a co-algebra structure Δ' defined in the same way. Show that Δ is a homomorphism of co-algebras. (You need to formulate the notion of maps between co-algebras which preserves the structure of co-algebras.)

6. Notation as in Problem 7 above. The polynomial ring $k[\underline{x}]$ is an $\mathbb{N}$-graded k-algebra. Let R be the graded dual of $k[\underline{x}]$, i.e. the direct sum of the dual of the graded pieces of $k[\underline{x}]$.

(i) Show that the co-algebra structure on $k[\underline{x}]$ gives R the structure of a commutative k-algebra.

(ii) The graded k-algebra $k[\underline{x}]$ has a k-basis $\underline{x}^I$, $I \in \mathbb{N}^n$, where $\underline{x}^I = x_1^{i_1} \cdots x_n^{i_n}$ if $I = (i_1, \ldots, i_n)$. Let $e_I \in R$, $I \in \mathbb{N}^n$ be the dual basis. Determine the product $e_I \cdot e_J$, i.e. express the product as a linear combination of e_K's, where $I, J \in \mathbb{N}^n$.

(iii) The algebra structure on $k[\underline{x}]$ defines a co-algebra structure

$$\mu^* : R \to R \otimes_k R$$

on R, where μ^* is the graded dual of the multiplication map

$$\mu : k[\underline{x}] \otimes_k k[\underline{x}] \to k[\underline{x}] \,.$$

Determine $\mu^*(e_I)$, i.e. express $\mu^*(e_I)$ as a linear combination of $e_J \otimes e_K$'s.

7. Do Problem 60 of Shatz-Gallier.