

MATH 602 ASSIGNMENT 5, FALL 2006

1. Let \mathbb{H} be the ring of Hamiltonian quaternions. (Recall that \mathbb{H} is a four-dimensional vector space over \mathbb{R} with basis $1, i, j, k$ satisfying the familiar identities.)

(i) Show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

(ii) Show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$.

(iii) Show that $M_n(\mathbb{H})^{\text{op}} \cong M_n(\mathbb{H})$ for every $n \geq 1$. (The superscript “op” denotes the *opposite ring*.)

2. Let V be a two-dimensional vector space over \mathbb{R} , and let $b : V \times V \rightarrow \mathbb{R}$ be a symmetric bilinear form over \mathbb{R} . Denote by $C(V, b)$ the quotient of the tensor algebra $\otimes_k^{\bullet}(V)$ by the ideal generated by elements of the form $v \otimes v - b(v, v)$.

(i) Suppose that b is positive definite. Show that $C(V, b)$ is isomorphic to $M_2(\mathbb{R})$.

(ii) Suppose that b is negative definite. Show that $C(V, b)$ is isomorphic to \mathbb{H} .

(iii) Suppose that b has signature $(1, 1)$, i.e. b is nondegenerate but not definite. Is $C(V, b)$ isomorphic to $M_2(\mathbb{R})$? Is $C(V, b)$ isomorphic to \mathbb{H} ?

3. Let k be a field. Let $A = k^{\mathbb{N}}$ be the product of countably many copies of k indexed by \mathbb{N} . In other words, A consists of infinite vectors (a_0, a_1, a_2, \dots) with entries in k ; addition and multiplication are performed coordinate-wise. Denote by I the ideal of A consisting of all elements (a_0, a_1, a_2, \dots) of A such that $a_i = 0$ for all but finitely many i 's.

(i) Show that I is not a finitely generated A -module.

(ii) Show that I is a projective A -module.

(iii) Show that $(A/I) \otimes_A I = (0)$.

(iv) Is I an A -linear direct summand of A ? In other words, is there an ideal J of A such that $I \cap J = (0)$ and $I + J = A$?

4. Let k be a commutative ring, $R = k[x_1, \dots, x_n]$ be the polynomial ring in variables x_1, \dots, x_n . Let S be the subring of symmetric polynomials in R . Show that R is a free S -module of rank $n!$.

5. Let $R = R_n$ be the polynomial ring $k[x_1, \dots, x_n]$ as in Problem 4 above. Let $s_1(\underline{x}), \dots, s_n(\underline{x})$ be the elementary symmetric polynomials in x_1, \dots, x_n . Let $p_i(\underline{x}) = \sum_{j=1}^n x_j^i$, $i = 1, \dots, n$. Let $r_n(Y_1, \dots, Y_n)$ be the polynomial in Y_1, \dots, Y_n such that $r_i(s_1, \dots, s_n) = p_i$. So we have polynomials $r_1(Y_1), r_2(Y_1, Y_2), \dots, r_n(Y_1, \dots, Y_n)$.

(1) For $n = 2, 3, 4$, compute the polynomials $r_n(\underline{Y})$.

(2) Prove that $r_i(s_1(\underline{x}), \dots, s_n(\underline{x})) = p_i(\underline{x})$ for $i = 1, \dots, n-1$.

(3) Prove that the monomials Y_1^n and $(-1)^{n-1} n Y_n$ appear in $r_n(\underline{Y})$.

(4) Show that

$$r_n(Y_1, \dots, Y_n) = \sum_{j=0}^{n-1} (-1)^j \begin{vmatrix} Y_{i+1} & Y_{i+2} & Y_{i+3} & \cdots & Y_{n-1} & Y_n \\ 1 & Y_1 & Y_2 & \cdots & Y_{n-i-2} & Y_{n-i-1} \\ 1 & 1 & Y_1 & \cdots & Y_{n-i-3} & Y_{n-i-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & Y_1 & Y_2 \\ 0 & 0 & 0 & \cdots & 1 & Y_1 \end{vmatrix}$$

6. Let A, B be matrices in $M_n(\mathbb{C})$.

(a) Show that A and B have the same characteristic polynomial if and only if $\text{Tr}(A^i) = \text{Tr}(B^i)$ for all $i \in \mathbb{N}$.

(b) Can you improve this criterion using a finite set of values of i ? If so, what is the minimal number of i 's you need?

7. Let k be a commutative ring. Let $D = \det(A)$, where A is the $n \times n$ -matrix whose (i, j) -entry is $p_{i+j-2} := \sum_{a=1}^n x_a^{i+j-2} \in k[x_1, \dots, x_n]$. Prove that $D = \prod_{1 \leq i < j \leq n} (x_i - x_j)^2$.