Math 602 Assignment 6, Fall 2006

1. Do Shatz-Gallier Problem 56, part 1 and 2.

2. Do Shatz-Gallier Problem 39, parts 1, 2, 3.

3. Let $A,\ B \in \mathrm{M}_2(\mathbb{Q})$. Define $A^{\mathrm{aug}},\ A^{\mathrm{aug}} \in \mathrm{M}_4(\mathbb{Q})$ by

$$A^{\mathrm{aug}} = \begin{pmatrix} 0 & I_2 \\ A & 0 \end{pmatrix}, \qquad B^{\mathrm{aug}} = \begin{pmatrix} 0 & I_2 \\ B & 0 \end{pmatrix},$$

   (i) Express the characteristic polynomial of A^{aug} in terms of the characteristic polynomial of A.

   (ii) Express the minimal polynomial of A^{aug} in terms of the minimal polynomial of A.

   (iii) Suppose that A^{aug} and B^{aug} are similar in $\mathrm{M}_4(\mathbb{Q})$, i.e. $\exists\, C \in \mathrm{GL}_4(\mathbb{Q})$ such that

$$C \cdot A^{\mathrm{aug}} \cdot C^{-1} = B^{\mathrm{aug}}.$$

   Either prove that A and B are similar in $\mathrm{M}_2(\mathbb{Q})$, or disprove this statement with a counter-example.

   (iv) (challenge problem) The construction of A^{aug} from A can be generalized to define $A^{\mathrm{aug}} \in \mathrm{M}_{2n}(\mathbb{Q})$ for any element $A \in \mathrm{M}_n(\mathbb{Q})$ in the obvious way. Redo parts (i)–(iii) above for arbitrary $n \in \mathbb{N}$. (Cf. Shatz-Gallier Problem 30.)

4. Let k be a commutative ring, and let M be a k-module.

   (i) Suppose that M is a free k-module of finite rank. Show that $\mathrm{T}^n\mathrm{S}(M)$ is naturally isomorphic to $\mathrm{Hom}_k(\mathrm{S}^n(M^\vee), k)$, where $M^\vee := \mathrm{Hom}_k(M, k)$. (You need to first define a natural map from $\mathrm{T}^n\mathrm{S}(M)$ to $\mathrm{Hom}_k(\mathrm{S}^n(M^\vee), k)$.)

   (ii) Suppose that M is a finitely generated projective k-module. Is the conclusion of (i) above still true?

   (iii) Suppose that M is a free k-module of infinite rank. Is the conclusion of (i) above still true?

Definition. Let k be a commutative ring, and let B be a commutative k-algebra.

(i) An *exponential sequence* in the k-algebra B is a sequence $\underline{a} = (a_n)_{n\in\mathbb{N}}$ such that $a_0 = 1$ and $a_m\, a_n = \binom{m+n}{m} a_{m+n}$ for all $m, n \in \mathbb{N}$. In other words, the formal power series $f(t) := \sum_{n\in\mathbb{N}} a_n t^n \in B[[t]]$ satisfies $f(t_1 + t_2) = f(t_1) + f(t_2)$. Denote by $\mathcal{E}(B)$ the set of all exponential sequences in B.

(ii) Let $\underline{a}, \underline{b} \in \mathcal{E}B$, $\lambda \in k$. Define $\underline{a} + \underline{b} \in \mathcal{E}(B)$, $\underline{a} \cdot \underline{b} \in \mathcal{E}(B)$ and $\lambda \cdot \underline{a} \in \mathcal{E}(B)$ as follows. If we write $\underline{a} + \underline{b} = (c_n)$, $\underline{a} \cdot \underline{b} = (d_n)$, and $\lambda \cdot \underline{a} = (e_n)$, then

$$c_n = \sum_{r+s=n} a_r b_s, \quad d_n = n!\, a_n\, b_n, \quad e_n = \lambda^n\, a_n \quad \forall n \in \mathbb{N}.$$

5. Notation as above.

(i) Show that the definition above gives $\mathcal{E}(B)$ the structure of a commutative k-algebra (not necessarily with 1).

(ii) Show that the construction $B \rightsquigarrow \mathcal{E}(B)$ is a functor from the category of commutative k algebra with 1 to the category commutative k-algebras not necessarily with 1. (See Shatz-Gallier pp. 39–40 for the definition of categories and functors.)

(iii) Suppose that k is a $\mathbb{Q}$-algebra. Show that for each $x \in B$, the sequence $\underline{a}$ with $a_n = \frac{1}{n!}x^n$ for all $n \in \mathbb{N}$ is an exponential sequence in B.

(iv) Assumption as in (ii). Show that the map $B \to \mathcal{E}(B)$ described in (ii) is a homomorphism of k-algebras.

(v) Let M be a free k-module. Show that for every element $x \in M$, the sequence $(\gamma_n(x))_{n\in\mathbb{N}}$ is an exponential sequence in $\mathrm{T}^\bullet\mathrm{S}(M)$, where $\gamma_n(x)) := x \otimes \cdots \otimes x \in \mathrm{T}^n\mathrm{S}(M)$.

Definition. Let k be a commutative algebra, and let M be a k-module. Let $F(M)$ be the the $\mathbb{N}$-graded free k-algebra with homogeneous generators (n, x) of degree n, where n runs through all natural numbers $n \in \mathbb{N}$ and x runs through all elements of M. Define a graded k-algebra with 1, denoted $\Gamma(M)$ as the quotient of $F(M)$ by the graded ideal generated by the following elements

- $(0, x) - 1$, $x \in M$,
- $(n, \lambda x) - \lambda^n(n, x)$, $n \in \mathbb{N}$, $\lambda \in k$, $x \in M$,
- $(n, x + y) - \sum_{n=r+s} (r, x)\,(s, y)$, $n \in \mathbb{N}$, $x, y \in M$,
- $(m, x)\,(n, x) = \binom{m+n}{m}\,(m + n, x)$.

Denote by $\Gamma_n(M)$ the degree n piece of the graded k-algebra $\Gamma(M)$. For each $n \in \mathbb{N}$ and each $x \in M$, denote by $\gamma_n(x)$ the image of (n, x) in $\Gamma_n(M)$.

6. Notation as above.

(i) Show that for each $x \in M$, the sequence $(\gamma_n(x))_{n\in\mathbb{N}}$ is an exponential sequence in $\Gamma(M)$.

(ii) Show that the map $\gamma : M \to \mathcal{E}(\Gamma(M))$ described in (i) is a k-module homomorphism.

(iii) (Universal property of $\Gamma(M)$) Suppose that B is a commutative k-algebra. Let $\phi : M \to \mathcal{E}(B)$ be a k-module homomorphism. Then there exists a unique k-algebra homomorphism $h : \Gamma(M) \to B$ such that $\phi = \mathcal{E}(h) \circ \gamma$, where $\mathcal{E}(h) : \mathcal{E}(\Gamma(M)) \to \mathcal{E}(B)$ is the map from $\mathcal{E}(\Gamma(M))$ to $\mathcal{E}(B)$ attached to h.

(iv) Reformulate (iii) in terms of adjoint functors. (See page 44 of Shatz-Gallier for the definition of adjoint functors.)

(v) Suppose that M is a free k-module. Show that $\Gamma(M)$ is isomorphic to $\mathrm{T}^\bullet\mathrm{S}(M)$ as commutative graded k-algebras.