

MATH 602 ASSIGNMENT 7, FALL 2006

1. Let  $R$  be a commutative ring. Let  $S$  be a subset of  $R$  which contains 1 but not 0, and closed under multiplication. Let  $M$  be an  $R$ -module such that  $S^{-1}M = (0)$ . Does there exist an element  $s \in S$  such that  $s \cdot m = 0$  for every element  $m \in M$ ? Either give a proof or a counter-example.
2. Let  $R$  be a commutative ring, and let  $I$  be maximal among non-principal ideals of  $R$ . Prove that  $I$  is a prime ideal.
3. Let  $R$  be an integral domain, and let  $S$  be a multiplicative closed subset of  $R$  which contains 1 but not 0. Assume that the localization  $S^{-1}R$  is integral over  $R$ , i.e. every element of  $S^{-1}R$  satisfies a monic polynomial relation with coefficients in  $R$ . Show that  $S^{-1}R = R$ .
4. Let  $R$  be a commutative ring, and let  $S$  be a multiplicative closed subset of  $R$  which contains 1 but not 0. Let  $R' = R[X_s : s \in S]$  be the free commutative  $R$ -algebra with  $S$  being its free generators. Denote by  $R_S$  the quotient of  $R'$  by the ideal  $I_S$  generated by all elements of the form  $sX_s - 1$ , where  $s$  runs through all elements of  $S$ . Let  $\alpha : R \rightarrow R_S$  be the composition of the canonical injection  $R \hookrightarrow R'$  with the quotient map  $R' \rightarrow R_S$ . Show that  $R_S$  is naturally isomorphic to  $S^{-1}R$ .
5. (a naive generalization of Gauss's lemma) Let  $R$  be a commutative ring. Let  $f, g \in R[x]$  be polynomials with coefficients in  $R$ . Assume that the ideal in  $R$  generated by the coefficients of  $f$  (resp.  $g$ ) is equal to  $R$ . So that the ideal generated by the coefficients of  $f \cdot g$  is equal to  $R$ .
6. Let  $R$  be a commutative ring.
  - (i) Show that an element  $a \in R$  is nilpotent (i.e. there exists  $m \in \mathbb{N}$  such that  $a^m = 0$ ) if and only if  $a$  belongs to every prime ideal of  $R$ .
  - (ii) Let  $A \in M_n(R)$  be an  $n \times n$  matrix with coefficients in  $R$ . Let  $f(T) = T^n + \sum_{i=1}^{n-1} a_i T^i$  be the characteristic polynomial of  $A$ . Prove that  $A$  is nilpotent (i.e. there exists  $m \in \mathbb{N}$  such that  $A^m = 0$ ) if and only if  $a_{n-1}, \dots, a_1, a_0$  are all nilpotent.
7. Let  $M$  be a finitely generated module over a commutative ring  $R$ . Prove that every surjective  $R$ -module homomorphism  $\alpha : M \rightarrow M$  is a bijection. (Hint: Use the proof of Nakayama's lemma in class.)
8. Let  $K$  be a field, and let  $R = K[x, y]$  be the polynomial ring over  $K$  in two variables. Suppose that  $I, J$  are two ideals of  $R$  such that  $I^2 = J^2$ . Is  $I$  necessarily equal to  $J$ ? Either give a proof or a counter-example.