

MATH 602 ASSIGNMENT 8, FALL 2006

1. Let  $R$  be a commutative ring.
  - (i) Let  $I_1, I_2$  be ideals of  $R$  such that  $R/I_1$  and  $R/I_2$  are Noetherian. Show that  $R/(I_1 \cap I_2)$  is Noetherian.
  - (ii) Generalize (i) to a finite number of ideals of  $R$
2. Let  $R$  be an integral domain, and let  $K$  be the field of fractions of  $R$ . For any non-zero ideal  $I \subseteq R$ , define  $I^{-1}$  be the  $R$ -submodule of  $K$  consisting of all elements  $x \in K$  such that  $xI \subseteq R$ . A non-zero ideal  $I \subseteq R$  is said to be *invertible* if  $I^{-1} \cdot I = R$ , where  $I^{-1} \cdot I$  is the ideal of  $R$  generated by all elements of the form  $xy$  with  $x \in I^{-1}$  and  $y \in I$ .
  - (i) Show that every invertible ideal  $I$  is finitely generated.
  - (ii) Suppose that  $R$  is a local ring. Show that every invertible ideal of  $R$  is principal.
  - (iii) Show that a non-zero finitely generated ideal  $I$  is invertible if and only if  $IR_{\mathfrak{m}}$  is an invertible ideal of  $R_{\mathfrak{m}}$  for every maximal ideal  $\mathfrak{m} \subset R$ .
  - (iv) (extra credit) Is the statement (iii) true without assuming that  $I$  is finitely generated? Either give a proof or a counter-example.
3. Let  $R$  be a commutative ring. Prove that the following statements are equivalent.
  - (i)  $R$  is a Jacobson ring.
  - (ii) Every radical ideal in  $R$  is an intersection of maximal ideals.
  - (iii) Every prime ideal in  $R$  is an intersection of maximal ideals.
  - (iv) Every Goldman ideal in  $R$  is an intersection of maximal ideals.
4. Let  $K$  be a field, and let  $L$  be an algebraically closed field containing  $K$ . Let
 
$$f_1(\underline{x}), \dots, f_m(\underline{x}) \in K[x_1, \dots, x_n]$$
 be polynomials with coefficients in  $K$ . Assume that the system of polynomial equations  $f_1 = \dots = f_m = 0$  has no solution in  $L^n$ . Show that there exists polynomials  $g_1(\underline{x}), \dots, g_m(\underline{x}) \in K[x_1, \dots, x_n]$  such that  $\sum_{i=1}^m g_i f_i = 1$ .
5. Let  $R \subset S$  be an inclusion of commutative rings. Assume that  $R$  be a Jacobson ring, and  $S$  is a finitely generated  $R$ -algebra. Prove that  $\mathfrak{m} \cap R$  is a maximal ideal of  $R$  for every maximal ideal  $\mathfrak{m}$  of  $S$ .
6. Let  $R$  be a commutative ring,  $f, g \in R[x]$  be polynomials with coefficients in  $R$ . Let  $I$  (resp.  $J$ , resp.  $K$ ) be the ideal generated by the coefficients of  $f$ ,  $g$ ,  $fg$  respectively.
  - (i) Show that  $K \subseteq IJ$ .
  - (ii) Either give an example such that  $K \neq IJ$ , or show that none exists.
  - (iii) If  $I = R$ , prove that  $J = K$ .
  - (iv) Suppose that  $R$  is an integral domain and  $I$  is invertible as defined in Problem 2. Show that  $IJ = K$ .