MATH 602 ASSIGNMENT 9, FALL 2006

Definition Let R be a ring with 1. Define the *radical* of R to be the intersection of all maximal left ideals of R. The above definitions uses left R-modules. When we want to emphasize that, we say that $\mathfrak{n}$ is the *left radical* of R.

Recall that R is said to be *semisimple* (or left semisimple, to be more precise) if the free left R-module underlying R is a sum of simple R-module; R is said to be *simple* (or left semisimple, to be more precise) if it is semisimple, and there is only one simple R-module up to isomorphism.

1. Let R be a ring with 1. Let $\mathfrak{n}$ be the radical of R

(i) Show that there exists a maximal left ideal in R. Deduce that the radical of R is a proper left ideal of R. (Hint: Use Zorn's Lemma.)

(ii) Show that $\mathfrak{n} \cdot M = (0)$ for every simple left R-module M. (Hint: Show that for every $0 \neq x \in M$, the set of all elements $y \in R$ such that $y \cdot x = 0$ is a maximal left ideal of R.)

(iv) Suppose that I is a left ideal of R such that $I \cdot M = (0)$ for every simple left R-module M. Prove that $I \subseteq \mathfrak{n}$.

(v) Show that $\mathfrak{n}$ is a two-sided ideal of R. (Hint: Use (iv).)

(vi) Let I be a left ideal of R such that $I^n = (0)$ for some positive integer n. Show that $I \subseteq \mathfrak{n}$.

(vi) Show that the radical of $R/\mathfrak{n}$ is zero.

2. Is $\mathbb{Z}$ a semisimple ring? Is the polynomial rings $\mathbb{Q}[x]$ and $\mathbb{Q}[x, y]$ semisimple? What are the radical of $\mathbb{Z}$, $\mathbb{Q}[x]$ and $\mathbb{Q}[x, y]$?

3. Let K be a field and let R be the subring of $\mathrm{M}_3(K)$ consisting of all upper-triangular matrices in R. Determine the radical of R.

4. Let K be a field. Show that the radical of $\mathrm{M}_n(K)$ is zero, for all $n \geq 1$.

5. Let K be a field containing $\mathbb{F}_p$, the finite field with p elements.

(i) Show that the group ring $K[\mathbb{Z}/p\mathbb{Z}]$ is not semisimple.

(ii) Let G be a finite group such that $|G| \equiv 0 \pmod{p}$. Show that $K[G]$ is not semisimple.

6. Let K be a field, and let V be a finite-dimensional vector space over K. Let S be a subset of $\mathrm{End}_K(V)$, and let $R = K[S] \subseteq \mathrm{End}_K(V)$ be the K-subalgebra of $\mathrm{End}_K(V)$ generated by S.

(i) Suppose that $S = \{T\}$, where T is an element of $\mathrm{End}_K(V)$. Show that $R = K[T]$ is semisimple if and only if T is semisimple, in the sense that the minimal polynomial of T is equal to the characteristic polynomial of T.

(ii) Can you relate the property that V is a semisimple R-module to the property that R is semisimple?

7. Let R be a ring with 1 and let $\mathfrak{n}$ be the (left) radical of R.

(i) Let $x \in \mathfrak{n}$. Show that $R \cdot (1+x) = R$, i.e. there exists an element $z \in R$ such that $z \cdot (1+x) = 1$.

(ii) Suppose that J is a left ideal of R such that $R \cdot (1+x) = R$ for every $x \in J$. Show that $J \subseteq \mathfrak{n}$. (Hint: If not, then there exists a maximal left ideal $\mathfrak{m}$ of R such that $J + \mathfrak{m} \ni 1$.)

(iii) Let $x \in \mathfrak{n}$, and let z be an element of R such that $z \cdot (1+x) = 1$. Show that $z - 1 \in \mathfrak{n}$. Conclude that $1 + \mathfrak{n} \subset R^{\times}$.

(iv) Show that the $\mathfrak{n}$ is equal to the right radical of R. (Hint: Use the analogue of (i)–(iii) for the right radical.)