Exercise 10, 12/06/2005

1. Let F be a local field, and denote by $\epsilon(\chi, \psi, \mu)$ the local ϵ-factor attached to a quasi-character χ of $F^\times$, a non-trivial character ψ of $(F, +)$, and a Haar measure μ for $(F, +)$. Prove that

(i) $\epsilon(\chi, \psi, c\mu) = c \cdot \epsilon(\chi, \psi, \mu)$ for every $c > 0$.

(ii) $\epsilon(\chi, x \mapsto \psi(ax), \mu) = \chi(a) \cdot |a|_F^{-1} \cdot \epsilon(\chi, \psi, \mu)$ for every $a \in F^\times$.

2. Let F be a global field, and let $\mu^\times$ be a Haar measure on $\mathbb{A}_F^\times$. Recall that for every standard function Φ on $\mathbb{A}_F$, the global zeta function is defined by

$$Z(\chi, \Phi) = Z(\chi, \Phi, \mu^\times) = \int_{\mathbb{A}_F^\times} \chi(x)\Phi(x) \cdot d\mu^\times(x)\,,$$

where χ varies in the one-dimensional complex manifold $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{A}_F^\times/F^\times, \mathbb{C}^\times)$ of all idele class quasi-characters for F. Determine the least upper bound $\sigma_0 \in \mathbb{R}$ such that $Z(\chi, \Phi)$ is absolutely convergent for all χ such that $\mathrm{Re}(\chi) > \sigma_0$ and all standard functions Φ.

3. Let F be a local field, let $\mu^\times$ be a Haar measure on $F^\times$. Recall that for every standard function ϕ on F, the local zeta function is defined by

$$Z(\chi, \phi) = Z(\chi, \phi, \mu^\times) = \int_{F^\times} \chi(x)\phi(x) \cdot d\mu^\times(x)\,,$$

where χ varies in the one-dimensional complex manifold $\mathrm{Hom}_{\mathrm{cont}}(F^\times, \mathbb{C}^\times)$ of quasi-characters of $F^\times$. Determine the least upper bound $\sigma_0 \in \mathbb{R}$ such that $Z(\chi, \phi)$ is absolutely convergent for all χ such that $\mathrm{Re}(\chi) > \sigma_0$ and all standard functions ϕ.

3. Notation as in question 2 above. Let χ be an idele class quasi-character of F Find a standard function Φ on $\mathbb{A}_F$ and a Haar measure $\mu^\times$ on $\mathbb{A}_F^\times$ such that $Z(\chi, \Phi, \mu^\times)$ is equal to the Hecke L-series $L(\chi)$ times a product of Γ-factors.

4. Show that the zeta function for $\mathbb{F}_q(t)$ is equal to $\frac{1}{(1-q^{-s})\cdot(1-q^{1-s})}$.

5*. Use the global functional equation to show that the zeta function for a global function field F is a rational function in q^{-s}, where q is the cardinality of the subfield of constants in F.

6. Let F be a global function field and let χ be an idele class quasi-character for F. Show that there exists an element $s \in \mathbb{C}$ such that $\chi \cdot \omega_{-s}$ is an idele class character of finite character, where $\omega_{-s}(\cdot) = |\cdot|_{\mathbb{A}_F}^{-s}$.

7. Determine which number fields have an idele class quasi-character which is not equal to the product of ω_s with an idele class character of finite order for any $s \in \mathbb{C}$.