Exercise 2, 9/12/2005

1. Let χ be a non-principal Dirichlet series modulo n, $n \in \mathbb{N}_{\geq 3}$. Show that

$$L(1,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n} = \prod_{p} \left(1 - \frac{\chi(p)}{p}\right)^{-1}$$

2. Let d be a positive integer. Find a formula for the number of primitive Dirichlet characters modulo d in terms of the prime factorization of d.

3. Let χ be a non-primitive Dirichlet character modulo d, $d \in \mathbb{N}_{\geq 3}$. Let χ^* be the primitive Dirichlet character equivalent to χ, i.e. χ^* is the primitive Dirichlet character with the property that $\chi^*(n) = \chi(n)$ for all $n \in \mathbb{Z}$ such that $(n,d) = 1$. Show that

$$\chi(n) = \sum_{\substack{rs=n \\ r|d}} \mu(r)\, \chi^*(r)\, \chi^*(s)\,,$$

where μ is the Möbius function.

4. Let p be an odd prime number, and let $\chi_p := \left(\frac{\cdot}{p}\right) : \mathbb{Z}/p\mathbb{Z} \to \mu_2$ be the quadratic residue character. Prove that

$$\tau(\chi_p) := \sum_{n=0}^{p-1} \chi_p(n)\, e^{2\pi\sqrt{-1}\,n/p} = \sum_{n=0}^{p-1} e^{2\pi\sqrt{-1}\,n^2/p}\,.$$

5. Let p be a prime number, and let χ be a non-trivial Dirichlet character modulo p. Denote by $\tau(\chi)$ the Gauss sum $\sum_{n=0}^{p} \chi(n)\, e^{2\pi\sqrt{-1}\,n/p}$. Show that $|\tau(x)| = \sqrt{p}$. (Hint: Consider $\tau(\chi) \cdot \overline{\tau(\chi)}$.)

*6. Notation as in Problem 5 above. Write $\tau(\chi) = \epsilon_\chi \cdot \sqrt{p}$. Prove that ϵ_χ is a root of unity if and only if χ is a real character.

7. Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of complex numbers. Let $f(x)$ be a continuously differentiable function on $\mathbb{R}_{>0}$. Define a function $A(x)$ on $\mathbb{R}$ by

$$A(x) = \sum_{n\leq x} a_n\,.$$

Show that

$$\sum_{1\leq n\leq x} a_n\, f(n) = A(x)\, f(x) - \int_1^x A(t)\, f'(t)\, dt$$

8. Show that the following statements are equivalent.

(i) $\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$, where $\pi(x) = \sum_{p\leq x} 1 \ \ \forall x \in \mathbb{R}$.

(ii) $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$, where $\theta(x) = \sum_{p\leq x} \log p \ \ \forall x \in \mathbb{R}$.

(iii) $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, where $\psi(x) = \sum_{1\leq n\leq x} \Lambda(n) \ \ \forall x \in \mathbb{R}$, and $\Lambda : \mathbb{N}_{\geq 1} \to \mathbb{N}$ is the function such that $\Lambda(n) = 0$ unless n is a power of a prime number, and $\Lambda(p^m) = \log p$ if p is a prime number.

(Hint: Use Problem 7.)

9. (a) Show that if $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, then $\lim_{x\to\infty} \frac{\psi_1(x)}{\frac{1}{2}x^2} = 1$, where $\psi_1(x) := \int_1^x \psi(t)\,dt$.

(b) Prove that conversely, if $\lim_{x\to\infty} \frac{\psi_1(x)}{\frac{1}{2}x^2} = 1$, then $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$. (Hint: We have

$$\frac{1}{x} \cdot \frac{\int_{\alpha x}^{x} \psi(t)\,dt}{\beta x - x} \leq \frac{\psi(x)}{x} \leq \frac{1}{x} \cdot \frac{\int_{x}^{\beta x} \psi(t)\,dt}{\beta x - x}$$

for all $0 < \alpha < 1$ and all $\beta > 1$.)

10. Suppose that $f(x)$ be a continuous function on $[1, \infty)$ such that $xf(x)$ is increasing and $\int_1^x f(t)\,dt \sim A \cdot x^m$ as $x \to \infty$, for some real numbers $A, m > 0$. Show that $f(x) \sim mA\,x^{m-1}$.

11. Consider the Dirichlet series $\sum_{n\geq 1} (-1)^{n-1}\, n^{-s}$.

(i) Show that this Dirichlet series converges for $\mathrm{Re}(s) > 0$. Let $\phi(s)$ be the holomorphic function on $\{\mathrm{Re}(s) > 0\}$ given by this Dirichlet series.

(ii) Relate $\phi(s)$ to $\zeta(s)$.

(iii) Show that $\phi(s) \neq 0$ for all $0 < s \leq 1$.

(iv) Conclude that $\zeta(s) \neq 0$ for all $0 < s < 1$.