Exercise 2A, 9/24/2005

Here is some hints for Problem 1 in Exercise set 2. Let χ be a non-trivial Dirichlet character.

1. Prove that

$$\sum_{n\le x} \frac{\chi(n)}{n}\log n = L(1,\chi)\cdot\left(\sum_{m\le x}\frac{\chi(m)}{m}\Lambda(m)\right) - R_1(x,\chi) - R_2(x,\chi),$$

where

$$R_1(x,\chi) = \left(\sum_{n>x}\frac{\chi(n)}{n}\right)\cdot\left(\sum_{m\le x}\frac{\chi(m)}{m}\Lambda(m)\right)$$

and

$$R_2(x,\chi) = \sum_{m\le x}\frac{\chi(m)}{m}\Lambda(m)\sum_{\frac{m}{x}<n\le x}\frac{\chi(n)}{n}.$$

2. Show that

$$|R_1(x,\chi)| = \mathrm{O}\left(\frac{\log x}{x}\right).$$

(Hint: Use Abel summation to estimate the two factors of $R_1(x,\chi)$.)

3. Show that

$$|R_2(x,\chi)| = \mathrm{O}\left(\frac{\psi(x)}{x}\right).$$

(Hint: Use Abel summation to estimate $\sum_{\frac{m}{x}<n\le x}\frac{\chi(n)}{n}$.)

4. Prove that there exists a constant C such that

$$\left|\sum_{m\le x}\frac{\chi(m)}{m}\Lambda(m)\right| \le C$$

for all $x>0$.

5. Show that the series

$$\sum_p \frac{\chi(p)}{p}$$

is convergent, where p runs through all prime numbers.
(Hint: Use the Abel summation formula.)