Exercise 3, 9/18/2005

This short set of problems provides a counter-example for Question 3 in Exercise 1.

1. Let $\sum_{n\geq 1} a_n\, n^{-s}$ be a Dirichlet series which is divergent for $s = 0$. Prove that the abscissa of convergence of this Dirichlet series is equal to

$$\limsup_{N\geq 1} \frac{\log|\sum_{n\leq N} a_n|}{\log N}.$$

2. For each $n \in \mathbb{N}_{\geq 1}$, let $d(n) = \sum_{d|n} 1$, the number of divisors of n. For instance, $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$. Prove that $\sum_{n\leq N} d(n) = N \log N + \mathrm{O}(N)$ as $N \to \infty$.
(Hint: The sum is equal to the number of integer points in the plane region

$$\left\{(x, y) \in \mathbb{R}^2 \,|\, 1 \leq x \leq N, 1 \leq y \leq \frac{N}{x}\right\}$$

Approximate the sum by an integral.)

3. Denote by $f(s)$ the Dirichlet series $\sum_{n\geq 1}(-1)^{n-1}n^{-s}$, and let $F(s) = f(s)^2$, the formal square of $f(s)$.

(i) Show that the abscissa of convergence of $f(s)$ is 0, and the abscissa of absolute convergence of $f(s)$ is 1. Moreover, $f(s)$ converges at $s = 0$.

(ii) Show that the abscissa of convergence of $F(s)$ and its abscissa of absolute convergence are both equal to 1.

(iii) Determine whether $F(s)$ is convergent at $s = 1$.