Exercise 3a, 10/9/2005

Correction to Exercise 3 and discussion of related issues

1. Comments on Problem 3, Exercise set 3.

(i) Statements (i), (ii) of are wrong. In fact the abscissa of convergence for $F(s)$ is at most $\frac{1}{2}$; see Problem 3 below.

(ii) There is a closed formula for the coefficients b_n of $F(s) = \left(\sum_{n\geq 1}(-1)^{n-1}n^{-s}\right)^2 = \sum_{n\geq 1} b_n\, n^{-s}$:

$$b_n = \begin{cases} d(n) & \text{if } n \text{ is odd} \\ -3d(n) + 4d(n/2) = (a-3)d(m) & \text{if } n = 2^a m,\ a \geq 1,\ m \text{ odd} \end{cases}$$

(iii) $\sum_{n \text{ odd}, n\leq N} d(n) = \frac{1}{4} N \log N + \mathrm{O}(N)$.

(iv) It would be interesting to determine the abscissa of convergence of $F(s)$.

2. Show that if $\sum_{n\geq 1} a_n\, n^{-s}$ is absolutely convergent at $s = s_0$ and $\sum_{n\geq 1} b_n\, n^{-s}$, then their formal product is also convergent at $s = s_0$.

3. Consider two Dirichlet series $f(s) = \sum_{n\geq 1} a_n\, n^{-s}$ and $g(s) = \sum_{n\geq 1} b_n\, n^{-s}$. Let

$$h(s) = \sum_{n\geq 1} u_n\, n^{-s}, \qquad u_n = \sum_{d|n} a_d\, u_{n/d} \quad \forall n \geq 1$$

be the formal product of $f(s)$ and $g(s)$. Let

$$A(n) = \sum_{m\leq n} a_n, \quad B(n) = \sum_{m\leq n} b_n, \quad U(N) = \sum_{n\leq N} u_n\,.$$

(i) Suppose that $f(s)$ converges at $s = s_1$ with $\mathrm{Re}(s_1) = \epsilon_1 > 0$ and $g(s)$ converges at $s = s_2$ with $\mathrm{Re}(s_2) = \epsilon_2 > 0$. Show that there exist positive numbers C_1, D_1, C_2, D_2 such that

$$|A_n| \leq C_1 n^{\epsilon_1},\ |a_n| \leq D_1 n^{\epsilon_1},\ |B_n| \leq C_2 n^{\epsilon_2},\ |b_n| \leq D_2 n^{\epsilon_2}\,.$$

(ii) Notation as in (i) above. Show that

$$|U(N)| \leq 2(C_1 D_2 + C_2 D_1) N^{\frac{1}{2}+\epsilon_1+\epsilon_2}$$

(iii) Assume that$f(s)$ and $g(s)$ are both convergent for $\mathrm{Re}(s) > 0$. Prove that $\sum_{n\geq 1} u_n\, n^{-s}$ is convergent for $\mathrm{Re}(s) > \frac{1}{2}$.

4. Let $f(s) = \sum_{n\geq 1} (-1)^{n-1} (\log 2n)^{-2}\, n^{-s}$, and let $F(s) = f(s)^2 = \sum_{n\geq 1} b_n\, n^{-s}$.

(i) Show that $f(s)$ is convergent at $s = 0$.

(ii) Find the abscissa of convergence and the abscissa of absolute convergence for $f(s)$

(iii) Show that $F(s)$ diverges at $s = 0$.
(Hint: Show that the b_n's are unbounded.)