Exercise 9, 11/10/2005

1. Determine explicitly the Pontriagin dual of $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

2. Determine the Pontiagin dual of $\widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

3. Identify $\mathbb{C}$ with its own Pontriagin dual via

$$(z, w) \mapsto \exp\left(2\pi\sqrt{-1}\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(zw)\right)$$

Determine the self-dual Haar measure on $\mathbb{C}$.

4. Identify $\mathbb{R}$ with its own Pontriagin dual via

$$(x, y) \mapsto \exp\left(2\pi\sqrt{-1}xy\right))$$

Determine the self-dual Haar measure on $\mathbb{R}$.

5. Determine explicitly the Pontriagin dual of ${\mathbb{Z}_p}^\times$ and ${\mathbb{Q}_p}^\times$.

6. Show that the Pontriagin dual of $\mathbb{A}_F$ is the restricted product $\prod'_v (F_v^*, \mathcal{O}_v^\perp)$.

7. Let G be a locally compact abelian group, μ be a Haar measure on G, and $\hat{\mu}$ be a Haar measure on the Pontriagin dual G^* of G such that

$$f(x) = \int_G \hat{f}(\hat{x})\overline{\langle x, \hat{x}\rangle}\, d\hat{\mu}(\hat{x})$$

for all $f \in C_c(G)$, where $\hat{f} = \mathcal{F}_\mu(f)$ is given by $\hat{f}(\hat{x}) = \int_G f(x)\langle x, \hat{x}\rangle\, d\mu(x)$. Let g be an element of $C_c(G)$, and let $\tilde{g} \in C_c(G)$ be given by $\tilde{g}(x) = \overline{g(-x)}$.

(i) Show that $\mathcal{F}_\mu(g * \tilde{g})(\hat{x}) = \hat{g}(\hat{x}) \cdot \overline{\hat{g}(\hat{x})}$.

(ii) Show that $||g||_\mu^2 = ||\hat{g}_{\hat{\mu}}||^2$. [Hint: Fourier inversion for $g * \tilde{g}$.]

(The point of this exercise is to tie the Plancherel theorem with Fourier inversion.)