MATH 602 HOMEWORK 1, FALL 2014

1. (a) Determine the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$.
(b) Compute $\mathrm{disc}\big(\mathbb{Q}(\sqrt[3]{2}/\mathbb{Q})\big)$ and $\mathscr{D}\big(\mathbb{Q}(\sqrt[3]{2}/\mathbb{Q})\big)$.

2. Let $F = \mathbb{Q}[T]/(T^3 + T + 1)$. Find the ring of integers $\mathscr{O}_F$ of this number field F and compute $\mathrm{disc}(F/\mathbb{Q})$.

3. Let p be an odd prime number. Let $\mathbb{Q}(\mu_p)$ be the cyclotomic field generated by a non-trivial p-th root of unity in $\mathbb{C}$.

(a) Show that $\mathbb{Q}(\mu_p)$ contains a unique quadratic subfield, i.e. a subfield of degree 2 over $\mathbb{Q}$.

(b) Prove that $\mathbb{Q}\left(\sqrt{\left(\frac{-1}{p}\right)\cdot p}\right)$ is the quadratic subfield in $\mathbb{Q}(\mu_p)$. Here $\left(\frac{?}{?}\right)$ is the Legendre symbol.

4. Describe/determine the group ${\mathbb{Q}_p}^\times/({\mathbb{Q}_p}^\times)^2$, where p is a prime number.

5. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. Denote by $|\ |_p$ the unique extension to $\overline{\mathbb{Q}}_p$ of the normalized absolute value of $\mathbb{Q}_p$. For every positive real number a, let $S_a = \{x \in \overline{\mathbb{Z}_p} : |x|_p < a$.

(a) Show that S_a is an ideal of $\overline{\mathbb{Z}_p}$ for every positive real number a.

(b) Show that the ideal S_a is *not* a finitely generated ideal of $\overline{\mathbb{Z}_p}$.

(c) Besides the ideals S_a's, are there other non-finitely generated ideals of $\overline{\mathbb{Z}_p}$? Either give such an example, or show that no such example exists.

6. Let N be a positive integer.

(a) Show that up to isomorphism there are only a finite number of extension fields of $\mathbb{Q}_p$ of degree at most N.

(b) Is it true that to isomorphism there are only a finite number of extension fields of $\mathbb{F}_p((t))$ of degree at most N?