MATH 602 HOMEWORK 2, FALL 2014

1. Is there a non-perfect field $k \supset \mathbb{F}_p$ such that $W(k)$ is a discrete valuation ring with p as a uniformizer? Either give an example of a non-perfeld field k which has this property, or show that no such non-perfield k exist.

2. Suppose that k is a locally compact field and G is an algebraic group over k of dimension d.

(a) Show that every left invariant non-trivial differential d-form α on G gives rise to a left-invariant Haar measure μ_α on $G(k)$, where $G(k)$ is the set of all k-points of G.

(b) Illustrate (a) in the following cases: $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$.

[An algebraic group over a field F is an algebraic variety G together with two morphisms, multiplication $\mu : G \times G \to G$ and inverse $\iota : G \to G$ satisfying the usual conditions for group laws. You can find definitions of *linear algebraic groups* such as Springer, Borel or Humphreys.]

3. Let p be an odd prime number, and let K be an unramified quadratic extension of $\mathbb{Q}_p$. Determine the group structure of $\mathscr{O}_K^\times$. In particular, is it isomorphic to a product of a finite group and a direct sum of copies of $\mathbb{Z}_p$?

4. (a) Is there a field automorphism of $\mathbb{Q}_p$ whose restriction to $\mathbb{Q}_p \cap \overline{\mathbb{Q}}$ is a non-trivial field automorphism of $\mathbb{Q}_p \cap \overline{\mathbb{Q}}$?

(b) Is there a field automorphism of $\mathbb{R}$ whose restriction to $\mathbb{R} \cap \overline{\mathbb{Q}}$ is a non-trivial field automorphism of $\mathbb{R} \cap \overline{\mathbb{Q}}$?