Math 602 Homework 3, Fall 2014

1. Let K be a global field. Let
$$j_1 : \mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K$$
be the inclusion map, and let
$$j : \mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K$$
be the map $j : x \mapsto (x, x^{-1})$ for every $x \in \mathbb{A}_K^\times$. Let $\mathfrak{T}_1$ be the topology on $\mathbb{A}_K^\times$ induced by the inclusion j_1; let $\mathfrak{T}$ be the topology induced on $\mathbb{A}_K^\times$ induced by j and the product topology on $\mathbb{A}_K \times \mathbb{A}_K$

(a) Clearly the topology $\mathfrak{T}_1$ is coarser than $\mathfrak{T}$. Are they equal?

(b) Show that $j^{-1}(C)$ is compact for every compact subset $C \subset \mathbb{A}_K \times \mathbb{A}_K$.

(c) Give an example of a compact subset $C_1 \subset \mathbb{A}_K$ such that $j_1^{-1}(C_1)$ is not compact.

2. Let K be a number field. Let $H(\mathbb{A}_K)$ be the subgroup of $\mathrm{GL}_3(\mathbb{A}_K)$ consisting of all 3×3 matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in \mathbb{A}_K$. Similarly let $H(K)$ be the subgroup of $\mathrm{GL}_3(K)$ consisting of all 3×3 matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in K$.

(a) Show that $H(K)$ is a discrete subgroup of $H(\mathbb{A}_K)$.

(b) Show that $H(K)\backslash H(\mathbb{A}_K)$ is compact.

3. (a) Consider the ring homomorphism $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \to \mathbb{A}_{\mathbb{Q},f}$ given by the inclusion map. Show that the associated ring homomorphism $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{A}_{\mathbb{Q},f}$ is an isomorphism.
(b) Consider the subgroup $\mathbb{R} \times \hat{\mathbb{Z}}$ of $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathbb{Q},f}$ and the associated map $f : (\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z} \to \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$. Is this map f a bijection? Either give a proof or a counter-example.

4. Let K be a number field.

(a) Let $K^\times \cdot \prod_{v \in \Sigma_{K,f}} \mathcal{O}_{K_v}^\times$ be the subgroup of $\mathbb{A}_{K,f}^\times = \prod'_{v \in \Sigma_{K,f}} K_v^\times$ generated by $K^\times$ and $\prod_{v \in \Sigma_{K,f}} \mathcal{O}_{K_v}^\times$. Show that $K^\times \cdot \prod_{v \in \Sigma_{K,f}} \mathcal{O}_{K_v}^\times$ is a *closed* subgroup of $\mathbb{A}_{K,f}^\times$.

(b) Assume that the subring of algebraic integers $\mathcal{O}_K$ in K is a principal ideal domain. Is $\mathbb{A}_{K,f}^\times$ equal to $K^\times \cdot \prod_{v \in \Sigma_{K,f}} \mathcal{O}_{K_v}^\times$? Either give a proof or give a counter-example.

5. Let K be a global field. Let $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a system of m polynomials in n variables $x_1, \ldots, x_n$. Let $V(K) = \{(a_1, \ldots, a_n) \in K^n \mid f_i(a_1, \ldots, a_n) = 0 \ \forall i = 1, \ldots, m\}$. Let $V(\mathbb{A}_K) = \{(a_1, \ldots, a_n) \in \mathbb{A}_K^n \mid f_i(a_1, \ldots, a_n) = 0 \ \forall i = 1, \ldots, m\}$, endowed with the subspace topology induced from the product topology of $\mathbb{A}_K^n$. Is $V(K)$ a *discrete* subset of $V(\mathbb{A}_K)$? Either give a proof or give a counter-example.

6. Let $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{A}_\mathbb{Q}$ and $ad - bc = 1$, endowed with the topology as a subspace of $\mathbb{A}_K^4$.

(a) Show that $\mathrm{SL}_2(\mathbb{Q})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$

(b) Show that $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q}) / \mathrm{SL}_2(\mathbb{Q})$ is *non-compact*.

(c) Show every left Haar measure on $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ is a right Haar measure.

(d) Show that $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q}) / \mathrm{SL}_2(\mathbb{Q})$ has *finite* measure for any Haar measure on $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$.