MATH 602 HOMEWORK 4, FALL 2014

1. Let D be a Dedekind domain and let K be the field of fractions of D. For every non-zero maximal ideal ideal $\wp$ of D, let $D_\wp$ be the localization of D at $\wp$, let $\hat{D}_\wp = \varprojlim_n D/\wp^n$ be the $\wp$-adic completion of D, and let $\hat{K}_\wp$ be the field of fractions of $\hat{D}_\wp$.

(a) Show that $\hat{D}_\wp$ is naturally isomorphic to the completion of the discrete valuation ring $D_\wp$ for every non-zero maximal $\wp$ of D.

(b) Let I, J be two non-zero D-submodules of K. Show that the following statements are equivalent.

(b1) $I \subseteq J$

(b2) $I \cdot D_\wp \subseteq J \cdot D_\wp$ as $D_\wp$-submodules of K, for every maximal ideal $\wp$ of D.

(b3) $I \cdot \hat{D}_\wp \subseteq J \cdot \hat{D}_\wp$ as $\hat{D}_\wp$-submodules of $\hat{K}_\wp$, for every maximal ideal $\wp$ of D. Here $I \cdot \hat{D}_\wp$ is the $\hat{D}_\wp$-submodule of $\hat{K}_\wp$ generated by I; similarly for $J \cdot \hat{D}_\wp$.

2. (This problem is a slightly more general version of an exercise given in class.) Let A be a Dedekind domain and let K be the fraction field of A. Let L be a finite separable extension of K and let B be the integral closure of A in L. Let $\mathscr{O}$ be an *order* in B, i.e. $\mathscr{O}$ is a subring of B which contains A and $\mathscr{O}$ contains a K-basis of L. (Consequently $B/\mathscr{O}$ is an A-module of finite length.) Let

$$\mathfrak{c}(\mathscr{O}) = \{x \in L \,|\, x \cdot B \subseteq \mathscr{O}\},$$

the conductor of the order $\mathscr{O}$, which was written as $(\mathscr{O} : B)$ in class. Let

$$\mathscr{D}^{-1}(B/A) = \{x \in L \,|\, \mathrm{Tr}_{L/K}(x \cdot B) \subset A\},$$

the inverse different of B/A. Let

$$\mathscr{D}^{-1}(\mathscr{O}/A) = \{x \in L \,|\, \mathrm{Tr}_{L/K}(x \cdot \mathscr{O}) \subset A\}.$$

(a) Show that $\mathfrak{c}(\mathscr{O})$ is the largest ideal of B which is contained in $\mathscr{O}$. (This was given in class as an exercise.)

(b) Prove that

$$\mathfrak{c}(\mathscr{O}) = \{x \in L \,|\, x \cdot \mathscr{D}^{-1}(\mathscr{O}/A) \subseteq \mathscr{D}^{-1}(B/A)\}.$$

(c) Suppose that $\alpha \in B$ is an element of B such that $L = K(\alpha)$ and let $f(T)$ be the minimal polynomial of α over K. Show that

$$\mathfrak{c}(A[\alpha]) = f'(\alpha) \cdot \mathscr{D}^{-1}(\mathscr{O}/A).$$

3. Let K be a number field. Consider $K^\times$ as a subgroup of $\mathbb{A}_{K,f}^\times$. Show that $K^\times \cdot \prod_{v \in \Sigma_{K,f}} \mathscr{O}_{K_v}^\times = \mathbb{A}_{K,f}^\times$ if and only if $\mathscr{O}_K$ is a principal ideal domain.

4. (a) For $K = \mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt[3]{5})$, determine whether $K^\times$ is discrete in $\mathbb{A}_{K,f}^\times$.

(b) Is $\mathrm{GL}_2(\mathbb{Q})$ discrete in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$?

(c) Show that for every open subgroup $U \subset \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q},f})$, $U \cap \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$.

(d) Is it true that every subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index in $\mathrm{SL}_2(\mathbb{Z})$ contains a subgroup of the form $U \cap \mathrm{SL}_2(\mathbb{Z})$ for some open subgroup $U \subset \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q},f})$? Either give a proof or a counter-example.