MATH 602 HOMEWORK 7, FALL 2014

1. Let K be a number field. Recall that Σ_K is the set of all places of K.

(a) Show that for every continuous character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$, there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\psi(\mathscr{O}_v) = 1$ for all $v \notin S$.

(b) Show that for every continuous character $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$, there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\chi(\mathscr{O}_v^\times) = 1$ for all $v \notin S$.

(c) Let F be a nonarchimedean locally compact field. Show that every continuous character $\psi : (F,+) \to \mathbb{C}^\times$ is unitary. Does this statement hold for archimedean locally compact fields?

(d) Let K be a global function field field. Show that every character of $\mathbb{A}_K \to \mathbb{C}^\times$ is unitary.

(f) Show that for every number field K, there exists a character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ which is *not* unitary.

2. Let K be a number field.

(1) Show that there is a natural bijection between (a) the set of all continuous characters of $\mathbb{A}_K$ and (b) the set of all sequences $(\psi_v)_{v\in\Sigma_K}$ indexed by Σ_K such that ψ_v is a character of K_v for each $v \in \Sigma_K$ and there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\psi_v(\mathscr{O}_v) = 1$ for every $v \notin S$.

(2) Show that there is a natural bijection between (a) the set of all continuous characters of $\mathbb{A}_K^\times$ and (b) the set of all sequences $(\chi_v)_{v\in\Sigma_K}$ indexed by Σ_K such that χ_v is a character of $K_v^\times$ for each $v \in \Sigma_K$ and there exists a finite subset $S \subset \Sigma_K$ containing $\Sigma_{K,\infty}$ such that $\chi_v(\mathscr{O}_v^\times) = 1$ for every $v \notin S$.

3. For each prime number p, let $\Lambda_p : \mathbb{Q}_p \to \mathbb{Z}[1/p]/\mathbb{Z}$ be the composition of the projection $\mathbb{Q}_p \to \mathbb{Q}_p/Zp$ with the inverse of the isomorphism $\mathbb{Q}_p/Zp \xrightarrow{\sim} \mathbb{Z}[1/p]/\mathbb{Z}$. Let $\Lambda_\infty : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the *negative* of the natural projection. Let $\psi_\mathbb{Q} : \mathbb{A}_\mathbb{Q} \to \mathbb{C}_1^\times$ be the map defined by

$$\psi_\mathbb{Q}(x) = \prod_{v\in\Sigma_\mathbb{Q}} e^{2\pi\sqrt{-1}\cdot\Lambda_v(x_v)} \quad \forall x = (x_v)_{v\in\Sigma_\mathbb{Q}} \in \mathbb{A}_\mathbb{Q}.$$

(a) For every finite extension field F of $\mathbb{Q}_p$, define $\psi_F : F \to \mathbb{C}^\times$ by

$$\psi_F(x) = e^{2\pi\sqrt{-1}\cdot\Lambda_p(\mathrm{Tr}_{F/\mathbb{Q}_p}(x))}.$$

Show ψ_F is a continuous additive unitary character of F, and that for every continuous additive character $\psi : F \to \mathbb{C}^\times$, there exists a unique element $y \in F$ such that $\psi(x) = \psi_F(xy)$ for every $x \in F$.

(b) Prove that $\psi_\mathbb{Q}$ is continuous and defines a non-trivial unitary character on $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$.

(c) For every number field K, define a homomorphism $\psi_K : \mathbb{A}_K \to \mathbb{C}_1^\times$ by $\psi_K(x) = \psi_\mathbb{Q} \circ \mathrm{Tr}_{K/\mathbb{Q}}(x)$. Show that for every $y \in \mathbb{A}_K$, the map $x \mapsto \psi_K(xy)$ is a unitary character of $\mathbb{A}_K$.

(d) Let K be a number field. Show that for every continuous unitary character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ on $\mathbb{A}_K$, there exists a unique element $y \in \mathbb{A}_K$ such that $\psi(x) = \psi_K(xy$ for all $x \in \mathbb{A}_K$.

(e) Show that for every number field K and every continuous character $\phi : \mathbb{A}_K/K \to \mathbb{C}^\times$, there exists a unique element $y \in K$ such that $\psi(x) = \psi_K(xy)$ for all $x \in \mathbb{A}_K$.

4. Find explicitly a Schwartz function f on $\mathbb{A}_{\mathbb{Q},f}$, an idele class character $\chi : \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times}$ and a Haar measure $d^{\times}x$ on $\mathbb{A}_{\mathbb{A},f}^{\times}$ such that the associated zeta function

$$\zeta(f, \chi \cdot \omega_s, d^{\times}x) = \int_{\mathbb{A}_{\mathbb{Q},f}^{\times}} f(x) \cdot (\chi \omega_s)(x)\, d^{\times}x$$

is equal to the Dirichlet L-function

$$L(s, \left(\tfrac{-1}{\cdot}\right)) = \sum_{p>2} \left(1 - \left(\tfrac{-1}{p}\right) p^{-s}\right)^{-1}$$

attached to the Legendre symbol $\left(\frac{-1}{\cdot}\right)$.