Exercise 1.

BASIC NOTIONS: SCHEMES, QUASI-COHERENT SHEAVES

1. (This and the following exercise, on the sheafification of a presheaf and the pull back of a sheaf, are meant to complement the Appendix on the theory of sheaves in Chap. I.) Let $\mathscr{F}$ be a presheaf on a topological space X. Show that there is a sheaf $a\mathscr{F}$ on X and a map $\mathscr{F} \xrightarrow{\alpha} a\mathscr{F}$ of presheaves such that α induces a bijection

$$\mathrm{Hom}(a\mathscr{F},\mathscr{G}) \longrightarrow \mathrm{Hom}(\mathscr{F},\mathscr{G})$$

for every sheaf $\mathscr{G}$ on X. Here the first Hom means maps in the category of sheaves, while the second Hom means maps in the category of presheaves (of sets).

[Hint: Try to use a direct limit construction to force the sheaf property hods. You probably will need to apply the same procedure twice, because when applied for the first time you are likely to get only a *separated presheaf*, i.e. for every open covering U_i of an open U, the map $\mathscr{G}(U) \longrightarrow \prod_i \mathscr{G}(U_i)$ is injective. Repeating the process, you get the exactness of

$$\mathscr{G}(U) \longrightarrow \prod_i \mathscr{G}(U_i) \rightrightarrows \prod_{i,j} \mathscr{G}(U_i \cap U_j)\,.]$$

2. Let $f\colon X \to Y$ be a continuous map of topological spaces.

   (i) Show that the functor f_* from the category of presheaves on X to the category of presheaves on Y has a left adjoint $f^\sharp$. [Hint: Let $\mathscr{F}$ be a sheaf of sets on Y. For any open subset $V \subset X$, let

   $$f^\sharp(U) = \varinjlim_{U \hookrightarrow f^{-1}(V)} \mathscr{F}(V)\,,$$

   where the indexing set of the direct limit is the set of all open subsets $V \subset Y$ such that $f(U) \subset V$.]

   (ii) Show that the functor f_* from the category of sheaves on X to the category of sheaves on Y has a left adjoint $f^\bullet$. [Hint: Let $f^\bullet\mathscr{F}$ be the sheafification $a\,f^\sharp(\mathscr{F})$ of the presheaf $f^\sharp(\mathscr{F})$.]

   (iii) When $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$, f is given by a ring homomorphism from S to R, and $\mathscr{F} = \tilde{M}$ is the quasi-coherent $\mathscr{O}_Y$ attached to an S-module M, check that $\mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_Y} f^*\mathscr{F}$ is naturally isomorphic to the quasi-coherent $\mathscr{O}_X$-module attached to $R \otimes_S M$.

3. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_Y$-module. Verify that $f^*\mathscr{F} := \mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_Y} f^\bullet\mathscr{F}$ is canonically isomorphic to the pull-back of quasi-coherent modules explained in Chap. I, after Cor. 5.6 and before 5.7. Similarly, suppose that $r\colon X \to S$ and $s\colon Y \to S$ are S-schemes, and $\mathscr{F}$ (resp. $\mathscr{G}$) is a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Verify that $p_1^*\mathscr{F} \otimes_{(s\circ p_2)^*\mathscr{O}_S} p_1^*\mathscr{G}$ is canonically isomorphic to the quasi-coherent $\mathscr{O}_{X\times_S Y}$-module "$\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_Y$" in Chap. I, after Cor. 5.6 and before 5.7.

4. (Complement to Prop. 3.11 of AG II) Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Show that $\mathscr{I} := \mathrm{Ker}(\mathscr{O}_Y \to f_*\mathscr{O}_X)$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_X$. (The closed subscheme of X defined by $\mathscr{I}$ is called the *schematic closure* of X in Y.)

5. Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of schemes, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Show that $f_*\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module. (Recall that f is quasi-separated means that the diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ is quasi-compact.)

6. Verify that for any commutative ring R with 1, the set of all R-valued points of $\mathrm{GL}_{n,\mathbb{Z}}$ is in bijection with the set of all units of the matrix algebra $\mathrm{M}_n(R)$.

7. Denote by $\mathbb{A}_{\mathbb{Q}}$ the ring of all $\mathbb{Q}$-adeles, defined to be the subset of $\mathbb{R} \times \prod_p \mathbb{Q}_p$, consisting of all sequences $(x_i)_{i \in \Sigma}$, where the indexing set Σ consists of ∞ and the set of all prime numbers, $x_\infty \in \mathbb{R}$, $x_p \in \mathbb{Q}_p$ for all p, and $x_p \in \mathbb{Z}_p$ for all but a finite number of p's. Describe explicitly the set of all $\mathbb{A}_{\mathbb{Q}}$-points of $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[T, 1/T])$, GL_n and $\mathbb{A}^1 - \{0, 1\}$.

8. Give an example of a sheaf on $\mathrm{Spec}(\mathbb{Z}[T])$ which is not quasi-coherent.

9. Let X be a scheme. Do infinite products exists in the category of all quasi-coherent $\mathscr{O}_X$ modules? Either give a proof or a counterexample.

10. Let k be a field. Are $\mathrm{Spec}(k[x,y,z]/(x^2 - y^2 - z^4))$ and $\mathrm{Spec}(k[x,y,z]/(x^2 - y^2 - z^2))$ isomorphic as k-schemes? Either give a proof or a counterexample.

11. Find all lines on the Fermat cubic surface in $\mathbb{P}^3$.

12. A *double six* in $\mathbb{P}^3$ is a pair of sextuples of disjoint lines $(l_1, \ldots, l_6)$, $(m_1, \ldots, m_6)$ such that $l_i \cap m_i = \emptyset$ for all i and l_i meets m_j at a point if $i \neq j$. Find a double six on the Fermat cubic. (Find the number of all double six's if you feel adventurous.)

13. Let k be a field of characteristic p. Let $\sigma\colon \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ be the morphism such that σ^* is the Frobenius homomorphism $u \mapsto u^p$ for $u \in k$. For any k-scheme X, denote by $X^{(p)}$ the fibre produce $X \times_{\mathrm{Spec}(k),\sigma} \mathrm{Spec}(k)$. Give an example in which the scheme $X^{(p)}$ is *not* isomorphic to X.

14. Give an example of an additive category which is not an abelian category.