Exercise 2

1. Work over a field k. Let $T \subset \mathbb{P}^2$ be the "triangle" defined by $x_0 x_1 x_2 = 0$, a closed subscheme. Let $f\colon \mathbb{P}^2 - T \to \mathbb{P}^2 - Z$ be the isomorphism defined in projective coordinates by

$$(x_0 : x_1 : x_2) \mapsto (\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}).$$

   Let Z be the Zariski closure of the graph of f in $\mathbb{P}^2 \times_{\mathrm{Spec}(k)} \mathbb{P}^2$, a closed subscheme of $\mathbb{P}^2 \times_{\mathrm{Spec}(k)} \mathbb{P}^2$. Let $\mathrm{pr}_1 : Z \to \mathbb{P}^2$ be the projection to the first factor of $\mathbb{P}^2 \times_{\mathrm{Spec}(k)} \mathbb{P}^2$, thought of as the source of the birational map f. Relate $\mathrm{pr}_1 : Z \to \mathbb{P}^2$ to a suitable blowing up of $\mathbb{P}^2$.

2. Give an example of a scheme X with two affine open subsets U and V such that $U \cap V$ is not affine.

3. Work over a base field k. Let X be a smooth quadric in $\mathbb{P}^3$, x_0 be a k-rational point of X, and let $X \overset{g}{\dashrightarrow} \mathbb{P}^2$ be the projection from x_0 to a plane disjoint from x_0, a rational map which is regular on $X - \{x_0\}$.

   (i) Show that g does not extend to a morphism on X.

   (ii) Show that g is a birational map.

   (iii) Determine all $\mathbb{P}^1$'s contracted by g.

   (iv) Let $\alpha\colon B \to X$ be the blowing up of X at x_0. Show that the birational map g induces a morphism $\beta\colon B \to \mathbb{P}^2$.

   (v) Let y_1 and y_2 be the image in $\mathbb{P}^2$ of the two lines in X contracted under g. Show that B is isomorphic to the blowing up of $\mathbb{P}^2$ at y_1 and y_2.

   (vi) Show that the birational map $\mathbb{P}^2 \overset{g^{-1}}{\dashrightarrow} X$ is given by the linear system of conics on $\mathbb{P}^2$ passing through y_1 and y_2.

   [Note: We have not defined the notion of a linear system so far. So you can leave this part aside and come back when you know what a linear system is.]

   (vii) Show that X is *not* isomorphic to the blowing up of $\mathbb{P}^2$ centered at a closed point.

4. Let k be a field, V be a finite dimensional vector space over k. Let $\rho\colon \mathrm{GL}_n \to \mathrm{GL}(V)$ be a k-linear rational representation of GL_n on V, i.e. the homomorphism ρ is a k-morphism of group schemes over k. Suppose that $v \in V$ is a vector fixed by the subgroup B of all upper-triangular elements in GL_n. Prove that v is fixed by GL_n.

   [Hint: The quotient variety GL_n / B is proper over k.]

5. Work over a field k. Let H be a hyperplane in $\mathbb{P}^n$, $n \geq 2$. Let $Z \subset H$ be a smooth hypersurface in H of degree d, $n \geq 2$. Let $f\colon X \to \mathbb{P}^n$ be the blowing of $\mathbb{P}^n$ with center Z. Let Y be the strict transform of H, i.e. Y is the closure in X of $f^{-1}(H - Z)$, where $H - Z$ denotes the complement of Z in H. By the universal property of blowing ups, the $\mathscr{O}_X$-module $\mathscr{J} := f^{-1}\mathscr{I}_Z \cdot \mathscr{O}_X$, or the ideal in $\mathscr{O}_X$ generated by the image of the sheaf of ideals $\mathscr{I}_{\mathscr{Z}} \subset \mathscr{O}_{\mathbb{P}^n}$ for $Z \subset \mathbb{P}^n$, is an invertible $\mathscr{O}_X$-module isomorphic to the sheaf "$\mathscr{O}_X(1)$" on $X = \mathscr{P}roj_{\mathbb{P}^n} \oplus_{n \geq 0} \mathscr{I}_Z^n$. Show that Y is isomorphic to H under the morphism f, and $\mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{O}_Y$ is isomorphic to $f^*\mathscr{O}_{\mathbb{P}^n}(-d) \otimes_{\mathscr{O}_X} \mathscr{O}_Y$.

6. Let X be a noetherian integral scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and let $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ be a global section of $\mathscr{L}^{\otimes n}$, $n \geq 2$. Let $B \subset X$ be the Cartier divisor defined by f, so that f defines an isomorphism $\mathscr{L}^{\otimes n} \cong \mathscr{O}_X(B)$. Let $\mathbf{L} := \mathcal{S}pec\left(\bigoplus_{m\geq 0} \mathscr{L}^{\otimes(-m)}\right) \xrightarrow{\tilde{\pi}} X$, thought of as the total space of the line bundle over X whose local sections is $\mathscr{L}$. Denote by T the tautological global section of $\tilde{\pi}^*\mathscr{L}$, corresponding to the canonical element

$$\mathbf{1} \in \Gamma(X, \mathscr{L}^{\otimes(-1)} \otimes \mathscr{L}) \subset \bigoplus_{m\geq 0} \Gamma(X, \mathscr{L}^{\otimes(-m)} \otimes \mathscr{L}) = \Gamma(\mathbf{L}, \tilde{\pi}^*\mathscr{L}).$$

The *cyclic cover* of order n of X attached to the triple $(X, \mathscr{L}, f)$ is by definition the divisor $Y \subset \mathbf{L}$ of the section $T^n - \pi^* f \in \Gamma(\mathbf{L}, \pi^*\mathscr{L}^{\otimes n})$. Let $\pi\colon Y \to X$ be the finite locally free morphism induced by $\tilde{\pi}$. Let $B_1 \subset Y$ be the Cartier divisor in Y attached to the $T|_Y \in \Gamma(Y, \pi^*\mathscr{L})$, the image in in $\Gamma(Y, \pi^*\mathscr{L})$ of the tautological section of $\tilde{\pi}^*\mathscr{L}$.

(i) Show that $\pi_*\mathscr{O}_Y$ is isomorphic to $\oplus_{0\leq m\leq n-1} \mathscr{L}^{\otimes(-m)}$ as an $\mathscr{O}_X$-module.

(ii) Verify that B_1 is the inverse image of B in Y, and we have a natural isomorphism $\pi^*\mathscr{L} \cong \mathscr{O}_Y(B_1)$. Consequently $\pi^*\mathscr{O}_X(B) \cong \mathscr{O}_Y(B_1)^{\otimes n}$.

7. Work over a field k of characteristic $\neq 2$. Let $B \subset \mathbb{P}^2$ be a smooth conic curve defined by a homogeneous quadratic polynomial $f(x,y,z)$. Let $\pi\colon Y \longrightarrow \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ attached to the triple $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1), f)$, a smooth projective surface.

(i) If l is a line in $\mathbb{P}^2$ meeting B at two distinct points, then $\pi^{-1}(l)$ is a smooth curve in Y and $\deg(\mathscr{L}|_{\pi^{-1}(l)}) = 2$.

(ii) If l is a tangent line to B, then $\pi^{-1}(l)$ is the union $\tilde{l}_1 \cup \tilde{l}_2$ of two smooth curves in Y meeting transversally at a point. Moreover $\deg(\mathscr{L}|_{\tilde{l}_i}) = 1$ for $i = 1, 2$.

(iii) Show that B is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

8. Let $X = F(a_1, \ldots, a_n) := \mathscr{P}roj_{\mathbb{P}^1} \mathrm{S}^\bullet\left(\mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathbb{P}^1(a_n)\right)$. Assume for simplicity that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $\pi\colon X \to \mathbb{P}^1$ be the structural morphism, so that X is a family of $\mathbb{P}^{n-1}$'s parametrized by $\mathbb{P}^1$. Denote by $\mathscr{O}_X(1)$ the universal invertible quotient $\mathscr{O}_X$-module of

$$\pi^*\left(\mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathbb{P}^1(a_n)\right).$$

For every local ring $(R, \mathfrak{m})$, let S_R be the set

$$\left\{ (t_0, t_1; x_1 : x_2 : \ldots : x_n) \in R^{n+2} \mid t_0 R + t_1 R = R, x_1 R + \cdots + x_n R = R \right\}$$

modulo the equivalence relation generated by

$$\begin{array}{lcll} (t_0, t_1; x_1 : x_2 : \ldots : x_n) & \sim & (t_0, t_1; \mu x_1 : x_2 : \ldots : \mu x_n) & \mu \in R^\times \\ (t_0, t_1; x_1 : x_2 : \ldots : x_n) & \sim & (\lambda t_0, \lambda t_1; \lambda^{-a_1} x_1 : \lambda^{-a_2} x_2 : \ldots : \lambda^{-a_n} x_n) & \lambda \in R^\times \end{array}$$

(i) Show that there is a functorial bijection, between $X(R)$ and the set S_R for every local ring $(R, \mathfrak{m})$.

(ii) Show by a counterexample that the above description of $X(R)$ is no longer valid when R is not a local ring.