Riemann Forms

Ching-Li Chai*

version 12/03/2014

The notion of *Riemann forms* began as a coordinate-free reformulation of the *Riemann period relations* for abelian functions discovered by Riemann. This concept has evolved with the progress in mathematics in the 150 years after Riemann. Nowadays it is often viewed from the perspective of abstract Hodge theory: a Riemann form gives an alternating pairing on a (pure) Hodge $\mathbb{Z}$-structure of Hodge type $\{(0,-1),(-1,0)\}$ and weight -1, with values in the "Tate twisted" version $\mathbb{Z}(1)$ of $\mathbb{Z}$.

Along more algebraic lines, the concept of Riemann forms was extended to abelian varieties over arbitrary base fields, after Weil [8] developed the theory of abelian varieties over an arbitrary base fields. For any (co)homology theory, for instance de Rham, Hodge, étale or crystalline cohomology theory, a Riemann form for an abelian variety is an alternating pairing induced by a *polarization* of A, on the first homology group $H_1(A)$ of an abelian variety A, with values in $H_1(\mathbb{G}_m)$, the first Tate twist of the homology of a point. A synopsis for the case of étale cohomology is given in §2.

§1. From Riemann matrices to Riemann forms

A version of theorem 1.1 appeared in [3, pp. 148–150], preceded by the high praise

> "il est extrèmement remarquable est c'est à M. le Dr. Riemann, de Göttingue, qu'on doit cette découverte analytique ..."

from Hermite. This theorem is stated in terms of period matrices for abelian functions in the article on *Riemann bilinear relations*; see [7, Ch. 5,§§9–11] for a classical treatment. The definition 1.2 of Riemann forms delivers the same conditions with a better perspective. The reader may consult [5, Ch. I §§2-3] and [4, II §3, III §6] for more information.

(1.1) THEOREM. *Let Λ be a lattice in a finite dimension complex vector space V. In other words Λ is a discrete free abelian subgroup of V whose rank is equal to $2\dim_{\mathbb{C}}(V)$. The compact complex torus X of the form V/Λ is isomorphic to (the $\mathbb{C}$-points of) a complex abelian variety if and only if the pair (V,Λ) admits a* **Riemann form**.

Recall that a complex abelian variety is a *complete* irreducible algebraic variety over $\mathbb{C}$ with an algebraic group law. The completeness condition here means that the set of all $\mathbb{C}$-points of this variety is compact under the complex topology. The existence of an algebraic group law implies that this variety can be embeded into a complex projective space as the locus of common zeros of a finite number of homogeneous polynomials.

*Partially supported by NSF grants DMS 1200271

(1.2) DEFINITION. Let V be a finite dimensional vector space over $\mathbb{C}$ of dimension $g \geq 1$, and let Λ be a lattice V, i.e. a discrete free abelian subgroup of V rank $2g$. A **Riemann form** (or a **polarization**) for (V,Λ) is a skew symmetric $\mathbb{Z}$-bilinear map

$$\mu : \Lambda \times \Lambda \to \mathbb{Z}$$

such that the map

$$(v_1, v_2) \mapsto \mu_{\mathbb{R}}(\sqrt{-1} \cdot v_1, v_2) \quad \forall v_1, v_2 \in V$$

is a symmetric positive definite $\mathbb{R}$-bilinear form on V, where $\mu_{\mathbb{R}} : V \times V \to \mathbb{R}$ is the unique skew-symmetric $\mathbb{R}$-bilinear which extends μ. Note that the last condition implies that

$$\mu_{\mathbb{R}}(\sqrt{-1} \cdot v_1, \sqrt{-1} \cdot v_2) = \mu_{\mathbb{R}}(v_1, v_2) \quad \forall v_1, v_2 \in V.$$

REMARK. The map $H : V \times V \to \mathbb{C}$ defined by

$$H(v_1, v_1) = \mu(\sqrt{-1} \cdot v_1, v_2) + \sqrt{-1} \cdot \mu(v_1, v_2) \quad \forall v_1, v_2 \in V,$$

is a *positive definite* Hermitian form whose imaginary part is equal to $\mu_{\mathbb{R}}$.[1]

(1.2.1) Example. Let S be a Riemann surface of genus $g \geq 1$. Regard $H_1(S,\mathbb{Z})$ as a lattice in $\Gamma(S,K_S)^{\vee} = \mathrm{Hom}_{\mathbb{C}}(\Gamma(S,K_S),\mathbb{C})$ via the injection

$$j(\gamma)(\omega) = \int_{\gamma} \omega \qquad \forall \gamma \in H_1(S,K_S),\ \forall \omega \in \Gamma(S,K_S)$$

Let $\frown : H_1(S,\mathbb{Z}) \times H_1(S,\mathbb{Z}) \longrightarrow \mathbb{Z}$ be the intersection product on S. Then $(-1)\cdot \frown$, the additive inverse of the intersection product, is a Riemann form for the lattice $H_1(S,\mathbb{Z})$ in $\Gamma(S,K_S)^{\vee}$.[2]

(1.2.2) DEFINITION. Let $\mu : V \times V \to \mathbb{Z}$ be a Riemann form of (V,Λ). A **canonical basis** for (Λ,μ) is a $\mathbb{Z}$-basis $v_1,\ldots,v_g,v_{g+1},\ldots,v_{2g}$ of Λ such that there exist positive integers $d_1,\ldots,d_g > 0$ with $d_i | d_{i+1}$ for $i = 1,\ldots,g-1$ and

$$\mu(v_i, v_j) = \begin{cases} d_i & \text{if } j = i+g \\ -d_i & \text{if } j = i-g \\ 0 & \text{if } j-i \neq \pm g \end{cases} \qquad \forall 1 \leq i, j \leq 2g.$$

The positive integers $d_1,\ldots,d_g$ are uniquely determined by μ, called the *elementary divisors* of μ. A polarization of a pair (V,Λ) is **principal** if all of its elementary divisors are equal to 1.

[1] Clearly μ and H determines each other. Some authors call H a Riemann form, focus more on the Hermitian form H instead of $\mathrm{Im}(H)$.

[2] This annoying sign is an unfortunate consequence of the choice of sign in the ususal definition of Riemann forms as given in 1.2. In many ways it would be more natural to require that $(v_1,v_2) \mapsto \mu(v_1, \sqrt{-1}\cdot v_2)$ be positive definite in 1.2, but a number of changes will be required if one adopts that.

Lemma 1.3 below gives a dictionary between the more traditional notion of Riemann matrices and the coordinate-free notion of Riemann forms. Lemma 1.3.1 shows that a Riemann form plus a choice of a *canonical basis* of Λ leads to a "normal form" for the Riemann matrix in terms of the Siegel upper-half space $\mathfrak{H}_g$. Thus the family of all Riemann forms with elementary divisors $d_1, \ldots, d_g$ on lattices in g-dimensional complex vector spaces are holomorphically parmetrized by the Siegel upper-half space $\mathfrak{H}_g$ up to integral symplectic transformations in $\mathrm{Sp}_{2g}(\mathbb{Z}^{2g}, J)$, where $J = J(d_1, \ldots, d_g)$ is the skew-symmetric pairing on $\mathbb{Z}^{2g}$ given by the matrix $\begin{pmatrix} 0_g & D \\ -D & 0_g \end{pmatrix}$ and D is the diagonal matrix with $d_1, \ldots, d_g$ along its diagonal.

(1.3) Lemma. *Suppose that μ is a Riemann form for (V, Λ). Let H be the positive definite Hermitian form on V with μ as its imaginary part. Let $v_1, \ldots, v_{2g}$ be a $\mathbb{Z}$-basis of Λ. Let E be the skew symmetric $2g \times 2g$ integer matrix whose (i, j)-th entry is $\mu(v_i, v_j))$ for all $i, j = 1, \ldots, 2g$. For every $\mathbb{C}$-basis $z_1, \ldots, z_g$ of V, the $g \times 2g$ matrix*

$$P = \left(z_r(v_j)\right)_{1 \le r \le g,\ 1 \le j \le 2g}$$

is a Riemann matrix with principal part E, i.e.

$$P \cdot E^{-1} \cdot {}^tP = 0_g, \quad \sqrt{-1} \cdot P \cdot E^{-1} \cdot {}^t\overline{P} > 0_g$$

Conversely every Riemann matrix P with a principle part E arises this way from a Riemann form on a pair (V, Λ).

(1.3.1) Lemma. *Suppose that μ is a polarization for (V, Λ) and $v_1, \ldots, v_{2g}$ is a canonical basis for (Λ, μ) with elementary divisors $d_1 | \cdots | d_g$ as in 1.2.2. Let $z_1, \ldots, z_g$ be the $\mathbb{C}$-linear function on V determined by $z_r(v_{g+j}) = \delta_{rj} \cdot d_j$ for all $r, j = 1, \ldots, g$. Then the Riemann matrix $P = \left(z_r(v_j)\right)_{1 \le r \le g,\ 1 \le j \le 2g}$ has the form*

$$P = (\Omega\, D)$$

for some element Ω in the Siegel upper-half space $\mathfrak{H}_g$, where D is the $g \times g$ diagonal matrix with entries $d_1, \ldots, d_g$ and the matrix $\Omega = \left(\Omega_{ij}\right)_{1 \le i, j \le g}$ is determined by

$$v_j = \sum_{i=1}^{g} \Omega_{ij} \cdot \frac{v_{g+i}}{d_i} \quad \text{for } j = 1, \ldots, g.$$

(1.3.2) Remark. In the context of 1.3.1, with the $\mathbb{Z}$-basis $v_1, \ldots, v_{2g}$ for Λ and the $\mathbb{C}$-coordinates $z_1, \ldots, z_g$ for V, the Hermitian form H on V becomes

$$(\vec{z}, \vec{w}) \mapsto {}^t\vec{z} \cdot \mathrm{Im}(\Omega)^{-1} \cdot \overline{\vec{w}}, \qquad \text{for } \vec{z}, \vec{w} \in \mathbb{C}^g.$$

and the matrix representation of the $\mathbb{R}$-linear endomorphism $J : v \mapsto \sqrt{-1} \cdot v$ of V for the $\mathbb{R}$-basis $v_1, \ldots, v_{2g}$ is

$$\begin{pmatrix} \mathrm{Im}(\Omega)^{-1} \cdot \mathrm{Re}(\Omega) & \mathrm{Im}(\Omega)^{-1} \cdot D \\ -D^{-1} \cdot \mathrm{Im}(\Omega) - D^{-1} \cdot \mathrm{Re}(\Omega) \cdot \mathrm{Im}(\Omega)^{-1} \cdot \mathrm{Re}(\Omega) & -D^{-1} \cdot \mathrm{Re}(\Omega) \cdot \mathrm{Im}(\Omega)^{-1} \cdot D \end{pmatrix}$$

(1.3.3) Remark. (1) The bilinear conditions for the Riemann matrix $(\Omega\, D)$ has an alternative equivalent form:

(1.3.3 a) $$\begin{pmatrix} I_g & -{}^t\Omega\cdot D^{-1}\end{pmatrix}\cdot\begin{pmatrix}0_g & D\\ -D & 0\end{pmatrix}\cdot\begin{pmatrix} I_g\\ -D^{-1}\cdot\Omega\end{pmatrix} = 0_g$$

(1.3.3 a) $$-\sqrt{-1}\cdot\begin{pmatrix} I_g & -{}^t\Omega\cdot D^{-1}\end{pmatrix}\cdot\begin{pmatrix}0_g & D\\ -D & 0\end{pmatrix}\cdot\begin{pmatrix} I_g\\ -D^{-1}\cdot\overline{\Omega}\end{pmatrix} > 0_g$$

(2) The columns of the matrix $\begin{pmatrix} I_g\\ -D^{-1}\cdot\overline{\Omega}\end{pmatrix}$ correspond to vectors

$$v_j\otimes 1-\sum_{j=1}^{g}\Omega_{ij}\cdot(v_{i+g}\otimes 1)\ \in(\Lambda\otimes_{\mathbb{Z}}\mathbb{C}),\quad j=1,\ldots,g,$$

which form a basis of the kernel of the $\mathbb{C}$-linear surjection

$$\mathrm{pr}: V\otimes_{\mathbb{R}}\mathbb{C}\twoheadrightarrow V,\qquad v_k\otimes 1\mapsto v_k\ \text{ for } k=1,\ldots,2g$$

So (1.3.3 a) says that Ker(pr) is an Lagrangian subspace for the alternating form $\mu\otimes\mathbb{C}$ on $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$.

(3) In terms of Hodge theory, Ker(pr) defines a Hodge $\mathbb{Z}$-structure with Hodge type $\{(0,-1),(-1,0)\}$ on Λ, such that Ker(pr) is equal to $\mathrm{Fil}^{0,-1}\subset\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$. Moreover $2\pi\sqrt{-1}\cdot\mathrm{Im}(H)$ defines a morphism $\Lambda\times\Lambda\to\mathbb{Z}(1)$ between Hodge $\mathbb{Z}$-structures, where $\mathbb{Z}(1)$ is the pure Hodge $\mathbb{Z}$-structure of type $\{(-1,-1)\}$ on the abelian group $2\pi\sqrt{-1}\subset\mathbb{C}$. The reader may consult [2] for a survey of modern Hodge theory.

§2. Algebraic incarnation of Riemann forms

(2.1) The étale version. We explain the definition the Riemann form attached to an ample divisor D of an abelian variety A over a field k, as an alternating pairing for the first étale homology group of A,[3] due to Weil[8]. Details can be found in [5, IV §20]. We assume for simplicity that the base field k is algebraically closed.

Let $H_1^{\mathrm{et}}(A) := \varprojlim_n A[n](k)$ and let $H_1^{\mathrm{et}}(\mathbb{G}_m) := \varprojlim_n \mathbb{G}_m[n](k)$, where n runs through all positive integers which are invertible in k. Here $A[n]$ denotes the group of all n-torsion points of A, and $\mathbb{G}_m[n](k)$ denotes the group of all n-th roots of 1 in k. The groups $H_1(A)$ and $H_1(\mathbb{G}_m)$ are naturally identified with the first étale homology groups of A and $\mathbb{G}_m$ respectively.

The Riemann form attached to the ample divisor D will be a bilinear pairing

$$E^D: H_1^{\mathrm{et}}(A)\times H_1^{\mathrm{et}}(A)\to H_1^{\mathrm{et}}(\mathbb{G}_m).$$

Given two elements $\underline{a}=(a_n)$ and $\underline{b}=(b_n)$ in $H_1^{\mathrm{et}}(A)$, represented as compatible systems of torsion points $a_n,b_n\in A[n]$, we need to produce a compatible system $\underline{c}=(c_n)$ of roots of unity in k.

[3] Any divisor on A algebraically equivalent to D will give the same alternating pairing.

(1) Since b_n is an n-torsion point of A, the divisors $[n]_A^{-1}(T_{-b_n}D) - [n]_A^{-1}(D)$ and $n \cdot (T_{-b_n}D) - n \cdot D$ are both principal by the theorem of the cube, where $T_{-b_n} : A \to A$ is the map "translation by $-b_n$". So there exist rational functions f_n, g_n on A such that the principal divisor (f_n), (g_n) are equal to $[n]_A^{-1}(T_{-b_n}D) - [n]_A^{-1}(D)$ and $n \cdot (T_{-b_n}D) - n \cdot D$ respectively.

(2) Because $[n]_A^*(g_n)$ and f_n^n have the same divisor, their ratio is a non-zero constant in k. Hence

$$T_{a_n}^*(f_n^n)/f_n^n = T_{a_n}^*[n]_A^*(g_n)/[n]_A^*(g_n) = 1.$$

Let

$$c_n := \frac{f_n}{T_{a_n}^*(f_n)} \in \mathbb{G}_m[n](k).$$

(3) One verifies that $c_{mn}^m = c_n$ for all $m, n \in \mathbb{N} \cap k^\times$, so the roots of unity c_n's produced in (2) form a compatible system $\underline{c} \in H_1^{\mathrm{et}}(\mathbb{G}_m)$. Define $E_{\mathrm{et}}^D(\underline{a}, \underline{b})$ to be this element $\underline{c} \in H_1^{\mathrm{et}}(\mathbb{G}_m)$.

(2.2) The general formalism for constructing Riemann forms. Suppose that H^* is a "good cohomology theory" for a suitable category of algebraic varieties. Examples of such good cohomology theories include Hodge, de Rham, étale and crystalline cohomologies. The following formal procedure produces an alternating pairing for any *polarization*[4] $\lambda : A \to {}^tA$ on an abelian variety A. For a reference for the crystalline version, see [1].

(1) Consider the Poincaré line bundle $\mathscr{P}$ on $A \times {}^tA$. The first Chern class $c_1(\mathscr{P})$ of $\mathscr{P}$ is an element of $H^2(A \times {}^tA)(1) \cong \mathrm{Hom}(H_1(A) \otimes H_1({}^tA), H_1(\mathbb{G}_m))$.

(2) The polarization λ of A induces a map $H_*(\lambda) : H_1(A) \to H_1({}^tA)$.

(3) Combining (1) and (2), one gets a bilinear map

$$(u, v) \mapsto c_1(\mathscr{P})(u, H_*(\lambda)(v))$$

from $H_1(A) \times H_1(A)$ to $H_1(\mathbb{G}_m)$, which turns out to be an alternating pairing. This is the Riemann form on $H_1(A)$ attached to λ.

References

[1] P. Berthelot, L. Breen and W. Messing, *Théorie de Dieudonné Cristalline II*, Lecture Notes in Math. 930, Springer, 1982.

[2] J.-L. Brylinski and S. Zucker, *An overview of recent advances in Hodge theory*, in *Several Complex Variables VI*, volume 69 of Encyclopaedia of Mathematical Sciences, Springer, 1997, pp.39–142.

[3] C. Hermite, *Note sur la théorie des fonctions elliptiques*, In *Œuvres De Charles Hermite, Tome II*, pp. 125–238. Extrait de la 6e edition du *Calcul différentiel et Calcul integral* de Lacroix, Paris, Mallet-Bachelier 1862.

[4] J. Igusa, *Theta Functions.* Springer-Verlag, 1972.

[4] Every ample divisor D on an abelian variety A gives rise to a polarization λ_D. It is the homomorphism from A to its dual abelian variety ${}^tA$, which sends any point $x \in A(k)$ to the isomorphism class of the $\mathscr{O}_A(T_{-x}D) \otimes_{\mathscr{O}_A} \mathscr{O}_A(D)^{\otimes -1}$. Note that the last invertible $\mathscr{O}_A$-module is algebraically equivalent to 0, hence corresponds to a point of ${}^tA$.

[5] D. Mumford, with Appendices by C. P. Ramanujam and Y. Manin. *Abelain Varieties.* 2nd ed., TIFR and Oxford Univ. Press, 1974. Corrected re-typeset reprint, TIFR and Amer. Math. Soc., 2008.

[6] B. Riemann, Theorie der Abel'schen Functionen. In *Bernhard Riemann's Gesammelte Mathematische Werke Und Wissenschaftlicher Nachlass*, 2nd ed., 1892 (reprinted by Dover, 1953), pp. 88–144; *J. reine angew. Math.* **54** (1857), pp. 115–155.

[7] C. L. Siegel, *Topics in Complex Function Theory, Vol. II, Abelian Functions and Modular Functions of Several Variables.* Wiley-Interscience, 1973.

[8] A. Weil, (a) *Sur les courbes algébriques et les variétés qui s'en déduisent*, Hermann, 1948. (b) *Variétés abéliennes et courbes algébriques*, Hermann, 1948. Second ed. of (a) and (b), under the collective title *Courbes algébriques et variétés abéliennes*, Hermann 1971.