Riemann's theta function

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Riemann's theta function $\theta(z,\Omega)$ was born in the famous memoir [13] on abelian functions. Its cousins, theta functions with characteristics $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega)$, are essentially translates of $\theta(z;\Omega)$. These theta functions can be viewed in several ways:

(a) They were first introduced and studied as holomorphic function in the z and/or the Ω variable.

(b) Geometrically these theta functions can be identified with sections of ample line bundles on abelian varieties and can be thought of as projective coordinates of abelian varieties. There values at the zero, known as *thetanullwerte*, give projective coordinates of moduli of abelian varieties.

(c) They are matrix coefficients, for the Schrödinger representation of the Heisenberg group and the oscillator (or Segal-Shale-Weil) representation of the metaplectic group.

The geometric point of view will be summarized in §1 and the group theoretic viewpoint in §2.

§1. Theta functions as sections of line bundles

For any positive integer g, denote by $\mathfrak{H}_g$ the Siegel upper-half space of genus g, consisting of all $g \times g$ symmetric complex matrices with positive definite imaginary part. We will use the following version of exponential function: $\mathbf{e}(z) := \exp(2\pi\sqrt{-1}\cdot z)$ for any $z \in \mathbb{C}$.

(1.1) Definition. (i) The **Riemann theta function** $\theta(z;\Omega)$ of genus g is the holomorphic function in two variables $(z,\Omega) \in \mathbb{C}^g \times \mathfrak{H}_g$, defined by the theta series

$$\theta(z;\Omega) := \sum_{m\in\mathbb{Z}^g} \mathbf{e}(\tfrac{1}{2}\,{}^t n\cdot\Omega\cdot n/2)\cdot\mathbf{e}({}^t n\cdot z).$$

(ii) Let $a,b \in \mathbb{R}^g$. The theta function $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z;\Omega)$ with **characteristics** a,b is the is the holomorphic function on $\mathbb{C}^g \times \mathfrak{H}_g$ defined by

$$\begin{aligned}\theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\Omega) &= \sum_{m\in\mathbb{Z}^g} \mathbf{e}(\tfrac{1}{2}\,{}^t(n+a)\cdot\Omega\cdot(n+a))\cdot\mathbf{e}({}^t(n+a)\cdot(z+b)) \\ &= \mathbf{e}(\tfrac{1}{2}\,{}^t a\cdot\Omega\cdot a + {}^t a\cdot(z+b))\cdot\theta(z+\Omega\cdot a+b;\Omega)\end{aligned}$$

Note that $\theta\left[\begin{smallmatrix} a+m \\ b+n \end{smallmatrix}\right](z;\Omega) = \mathbf{e}({}^t a\cdot n)\cdot\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z;\Omega)$ for all $m,n \in \mathbb{Z}^g$, so that $\mathbf{e}({}^t a\cdot b)\cdot\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z;\Omega)$ depends only on $a,b \bmod \mathbb{Z}^g$.

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The theta function $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z;\Omega)$ satisfies the following functional equation

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+\Omega\cdot m+n;\Omega) = \mathbf{e}({}^t a\cdot n - {}^t b\cdot m)\cdot \mathbf{e}(-\tfrac{1}{2}\,{}^t m\cdot\Omega\cdot m - {}^t m\cdot z)\cdot\theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\Omega) \quad \forall\, m,n\in\mathbb{Z}^g.$$

For any $a,b\in\mathbb{R}^g$, the family of holomorphic functions

$$u^{a,b}_{\Omega\cdot m+n}(z) := \mathbf{e}({}^t a\cdot n - {}^t b\cdot m)\cdot \mathbf{e}(-\tfrac{1}{2}\,{}^t m\cdot\Omega\cdot m - {}^t m\cdot z) \qquad m,n\in\mathbb{Z}^g$$

on V with values in $\mathbb{C}^\times$ forms a 1-cocycle $\mu^{a,b}$ for the lattice $\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g$ in $\mathbb{C}^g$, in the sense that

$$\mu^{a,b}_{\xi_1}(z)\cdot\mu^{a,b}_{\xi_2}(z+\xi_1) = \mu^{a,b}_{\xi_1+\xi_2}(z) \qquad \forall\,\xi_1,\xi_2\in\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g.$$

This 1-cocyle defines a line bundle on the compact complex torus $\mathbb{C}^g/(\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g)$ so that $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z;\Omega)$ can be interpreted as a section of this line bundle.

(1.2) THEOREM. (APPELL-HUMBERT) *Let Λ be a lattice in a finite dimensional complex vector space V.*

(1) *Every holomorphic line bundle on V/Λ is isomorphic to the quotient of the trivial line bundle on V via a 1-cocycle $\xi\to u_\xi(z)$ for Λ, where $u_\xi(z)$ is an entire function on V with values in $\mathbb{C}^\times$ for every $\xi\in\Lambda$ and $u_{\xi_1+\xi_2}(z) = u_{\xi_1}(z)\cdot u_{\xi_2}(z+\xi_1)$ for all $\xi_1,\xi_2\in\Lambda$.*

(2) *For any 1-cocycle $u_\xi(z)$ as in (1), there exists a quadruple (H,S,ℓ,ψ), where*

(2a) *$H: V\times V\to\mathbb{C}$ is a Hermitian form, conjugate linear in the second argument, such that $\mathrm{Im}(H)$ is $\mathbb{Z}$ valued on Λ, and $\psi:\Lambda\to\mathbb{C}_1^\times$ is a complex function with absolute values 1 such that $\psi(\xi_1+\xi_2)\cdot\psi(\xi_1)^{-1}\cdot\psi(\xi_2)^{-1} = (-1)^{\mathrm{Im}(H)(\xi_1,\xi_2)}$ for all $\xi_1,\xi_2\in\Lambda$,*

(2b) *$S: V\times V\to\mathbb{C}$ is a symmetric $\mathbb{C}$-bilinear form,*

(2c) *$\ell: V\to\mathbb{C}$ is a $\mathbb{C}$-linear function, and*

(2d) *$\psi:\Lambda\to\mathbb{C}_1^\times$ is a complex function with absolute values 1 such that*

$$\psi(\xi_1+\xi_2)\cdot\psi(\xi_1)^{-1}\cdot\psi(\xi_2)^{-1} = (-1)^{\mathrm{Im}(H)(\xi_1,\xi_2)} \qquad \forall\,\xi_1,\xi_2\in\Lambda,$$

such that

$$u_\xi(z) = \mathbf{e}\big(\tfrac{1}{2\sqrt{-1}}\cdot(H(z,\xi)+S(z,\xi))\big)\cdot\mathbf{e}\big(\tfrac{1}{4\sqrt{-1}}\cdot(H(\xi,\xi)+S(\xi,\xi))\big)\cdot\mathbf{e}(\ell(\xi))\cdot\psi(\xi) \quad \forall\,\xi\in\Lambda.$$

The quadruple (H,S,ℓ,ψ) is uniquely determined by the 1-cocycle $u_\xi(z)$. Conversely every quadruple (H,S,ℓ,ψ) satisfying conditions (2a)–(2d) determines a 1-cocyle for (V,Λ).

(3) *Let $\mathscr{L}$ and $\mathscr{L}'$ be two line bundles attached to two quadruples (H,S,ℓ,ψ) and (H',S',ℓ',ψ') as in (2).*

(3a) *$\mathscr{L}$ is isomorphic to $\mathscr{L}'$ if and only if $H=H'$ and $\psi=\psi'$.*

(3b) *$\mathscr{L}$ is algebraically equivalent to $\mathscr{L}'$ if and only if $H=H'$.*

(3c) *$\mathscr{L}$ is ample if and only if $\mathrm{Im}(H)$ is a polarization of (V,Λ), i.e. H is positive definite.*

(1.3) It is explained in the article on Riemann forms that for any polarization μ on a compact complex torus V/Λ for any choice of canonical $\mathbb{Z}$-basis $v_1,\dots,v_{2g}$ of Λ with elementary divisors $d_1|\cdots|d_g$, there exists a unique element $\Omega\in\mathfrak{H}_g$ such that the $\mathbb{C}$-linear map from V to $\mathbb{C}^g$ which sends v_i to d_i times the i-th standard basis of $\mathbb{C}^g$ induces a biholomorphic isomorphism from V/Λ to $\mathbb{C}^g/(\Omega\cdot\mathbb{Z}^g+D\cdot\mathbb{Z}^g)$, where D is the $g\times g$ diagonal matrix with $d_1,\dots,d_g$ as diagonal entries. Under this isomorphism, the polarization μ becomes the alternating pairing

$$\mu_D:(\Omega\cdot m_1+n_1,\Omega\cdot m_2+n_2)\mapsto {}^t m_1\cdot D\cdot n_2-{}^t n_1\cdot D\cdot m_2\qquad\forall m_1,n_2,m_2,n_2\in\mathbb{Z}^g$$

on the lattice $\Lambda_{\Omega,D}:=\Omega\cdot\mathbb{Z}^g+D\cdot\mathbb{Z}^g$, and the Hermitian form on $\mathbb{C}^g$ whose imaginary part is the $\mathbb{R}$-bilinear extension of μ_D is given by the formula

$$H_\Omega:(z,w)\mapsto {}^t z\cdot\mathrm{Im}(\Omega)^{-1}\cdot\overline{w}'\quad\forall z,w\in\mathbb{C}^g.$$

(1.3.1) An easy calculation shows that for any given $a,b\in\mathbb{R}^g$, the restriction to $\Lambda(\Omega,D)$ of the 1-cocycle $u^{a,b}$ corresponds to the quadruple $(H_\Omega,S_\Omega,0,\psi_0\cdot\chi_D^{a,b})$, where S_Ω is the symmetric $\mathbb{C}$-bilinear form

$$S_\Omega:\ (z,w)\mapsto -{}^t\cdot\mathrm{Im}(\Omega)^{-1}\cdot w\qquad\forall z,w\mathbb{C}^g,$$

or equivalently S_Ω is the unique CC-bilinear form on $\mathbb{C}^g$ which coincides with $-H_{\Omega,D}$ on the subset $\mathbb{C}^g\times(D\cdot\mathbb{Z}^g)\subset\mathbb{C}^g\times\mathbb{C}^g$, ψ_0 is the quadratic unitary character on $\Lambda_{\Omega,D}$ defined by

$$\psi_0(\Omega\cdot m+D\cdot n)=(-1)^{{}^t m\cdot n}\qquad\forall m,n\in\mathbb{Z}^g,$$

and $\chi_D^{a,b}$ is the unitary character on $\Lambda(\Omega,D)$ defined by

$$\chi_D^{a,b}(\Omega\cdot m+D\cdot n)=\mathbf{e}({}^t a\cdot D\cdot n-{}^t b\cdot m)\qquad\forall m,n\in\mathbb{Z}^g$$

Note that $\chi_D^{a,b}=\chi_D^{a',b'}$ if and only if $a-a'\in D^{-1}\cdot\mathbb{Z}^g$ and $b-b'\in\mathbb{Z}^g$. Let $\mathscr{L}_{\Omega,D}^{a,b}$ be the line bundle on $\mathbb{C}^g/\Lambda_{\Omega,D}$ given by the restriction to $\Lambda_{\Omega,D}$ of the 1-cocyle $u^{a,b}$. Clearly for every $a'\in a+D^{-1}\cdot\mathbb{Z}^g$, $\theta\left[\begin{smallmatrix}a'\\b\end{smallmatrix}\right](z,\Omega)$ defines a section of the line bundle $\mathscr{L}_{\Omega,D}^{a,b}$.

More generally, for every positive integer r, every $a'\in a+D^{-1}\cdot\mathbb{Z}^g$ and every $b'\in r^{-1}b+r^{-1}\mathbb{Z}^g$, $\theta\left[\begin{smallmatrix}a'\\b'\end{smallmatrix}\right](z,r^{-1}\Omega)$ defines a section of the line bundle $(\mathscr{L}_{\Omega,D}^{a,b})^{\otimes r}$. Underlying this statement is the fact that the pull-back of $\mathscr{L}_{r^{-1}\Omega,D}^{a',b'}$ via the isogeny $\mathbb{C}^g/\Lambda_{\Omega,D}\to\mathbb{C}^g/\Lambda_{r^{-1}\Omega,D}$ of degree r^g is isomorphic to $(\mathscr{L}_{\Omega,D}^{a,b})^{\otimes r}$, for every a',b' as above.

(1.3.2) PROPOSITION. (1) *For every line bundle $\mathscr{L}$ on the compact complex torus $\mathbb{C}^g/\Lambda_{\Omega,D}$ with polarization μ_D, there exists $a,b\in\mathbb{R}^g$ such that $\mathscr{L}$ is isomorphic to $\mathscr{L}_{\Omega,D}^{a,b}$. Moreover for any $a',b'\in\mathbb{R}^g$, $\mathscr{L}_{\Omega,D}^{a,b}$ is isomorphic to $\mathscr{L}_{\Omega,D}^{a',b'}$ if and only if $a'\in a+D^{-1}\cdot\mathbb{Z}^g$ and $b'\in b+\mathbb{Z}^g$.*

(2) $\dim_{\mathbb{C}}\Gamma\big(\mathbb{C}^g/\Lambda_{\Omega,D},\mathscr{L}_{\Omega,D}^{a,b}\big)=\det(D)=\prod_{i=1}^g d_i$. *Moreover as a' runs through a set of representatives of $(a+D^{-1}\cdot\mathbb{Z}^g)/\mathbb{Z}^g$, the theta functions $\theta\left[\begin{smallmatrix}a'\\b\end{smallmatrix}\right](z,\Omega)$ give rise to a $\mathbb{C}$-basis of $\Gamma(\mathscr{L}_{\Omega,D}^{a,b})$.*

(3) *For every positive integer r, we have* $\dim_{\mathbb{C}}\Gamma\big(\mathbb{C}^g/\Lambda_{\Omega,D},(\mathscr{L}^{a,b}_{\Omega,D})^{\otimes r}\big)=r^g\cdot\prod_{i=1}^g d_i$. *Moreover as* a' *runs through a set of representatives of* $(a+D^{-1}\cdot\mathbb{Z}^g)/\mathbb{Z}^g$ *and* b' *runs through a set of representatives of* $(r^{-1}b+r^{-1}\mathbb{Z}^g)/\mathbb{Z}^g$, *the global sections corresponding to* $\theta\left[\begin{smallmatrix}a'\\b'\end{smallmatrix}\right](z,r^{-1}\Omega)$ *form a* $\mathbb{C}$*-basis of* $\Gamma\big((\mathscr{L}^{a,b}_{\Omega,D})^{\otimes r}\big)$.

(1.4) THEOREM. *Let r be a positive integer,* $r\geq 3$. *For given* $a,b\in\mathbb{R}^g$, *let* $\{a_k|1\leq k\leq r^g\}$ *be a system of representatives of* $(a+D^{-1}\cdot\mathbb{Z}^g)/\mathbb{Z}^g$ *and let* $\{b_l|1\leq l\leq\prod_{i=1}^g d_i\}$ *be a system of representatives of* $(r^{-1}b+r^{-1}\mathbb{Z}^g)/\mathbb{Z}^g$. *The family of functions*

$$\Big\{\theta\left[\begin{smallmatrix}a_k\\b_l\end{smallmatrix}\right](z,r^{-1}\Omega)\colon\ 1\leq k\leq r^g,\ 1\leq l\leq\textstyle\prod_{i=1}^g d_i\Big\}$$

defines a projective embedding $\mathbb{C}^g/\Lambda_{\Omega,D}\hookrightarrow\mathbb{P}^{(\prod_{i=1}^g d_i)\cdot r^g-1}$ *of the compact complex torus* $\mathbb{C}^g/\Lambda_{\Omega,D}$. *In particular the field of abelian functions for the pair* $(\mathbb{C}^g,\Lambda_{\Omega,D})$ *is generated by the following family of quotients of theta functions*

$$\theta\left[\begin{smallmatrix}a_{k_1}\\b_{l_1}\end{smallmatrix}\right](z,r^{-1}\Omega)\Big/\theta\left[\begin{smallmatrix}a_{k_2}\\b_{l_2}\end{smallmatrix}\right](z,r^{-1}\Omega)\qquad 1\leq k_1,k_2\leq r^g,\ 1\leq l_1,l_2\leq\textstyle\prod_{i=1}^g d_i.$$

(1.5) The Riemann theta function was defined in §17 of Riemann's famous memoir [13] on abelian function. The notion of theta functions $\theta[\begin{smallmatrix}a\\b\end{smallmatrix}](z;\Omega)$ with characteristics is due to Prym [12, p. 25]. The second part of theorem 1.4 is a version of the "*theta theorem*" stated by Riemann in §26 of [13] and proved by Poincaré; see Siegel's formulation and comments in [17, p. 91]. Theorem 1.2 was proved by Appell and Humbert for abelian surfaces, called "hyperelliptic surfaces" at their time. The higher dimensional cases of theorem 1.2 and theorem 1.4 are due to Lefschetz. For more information related to this section, we recommend [9, Ch. I & Ch. III §17], [10, Ch. II, §§1–1], [2, Ch. II–III] and [5, Ch. I–II].

Theta functions in dimension $g=1$ go back to Jacobi, who obtained their properties by algebraic methods through his theory of elliptic functions; see [3] and [4].[1] We refer to [20, Ch. XXI] for a classical treatment and [20, pp. 462–463] for historical comments on theta functions, which appeared first in Euler's investigation on the *partition function* $\prod_{n=1}^n(1-x^nz)^{-1}$.

§2. Theta functions as matrix coefficients

Historically the infinite dimensional representations closely related to theta functions was inspired by quantum mechanics. First one has the Schrödinger representation, an irreducible unitary projective representation of $\mathbb{R}^{2g}$. There is also a projective representation of the symplectic group $\mathrm{Sp}(2g,\mathbb{R})$, on the space underlying the Schrödinger representation, called the oscillator representation. When viewed in the z-variable, the Riemann theta function $\theta(z;\Omega)$, is essentially a matrix

[1] The theta functions $\theta_1(z,q)$, $\theta_2(z,q)$ $\theta_3(z,q)$, $\theta_4(z,q)$ in Jacobi's notion [4] [20, Ch. XXI], where $z\in\mathbb{C}$ and $q=\mathbf{e}(\tau/2)$, $\tau\in\mathfrak{H}_1$, are equal to: $-\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z/\pi;\tau)$, $\theta[\begin{smallmatrix}1/2\\0\end{smallmatrix}](z/\pi;\tau)$, $\theta[\begin{smallmatrix}0\\0\end{smallmatrix}](z/\pi;\tau)$ and $\theta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right](z/\pi;\tau)$ respectively. In Jacobi's earlier notation in [3] are four theta functions $\Theta(u,q),\Theta_1(u,q),\mathrm{H}(u,q)$ and $\mathrm{H}_1(u,q)$, where the symbol H denotes capital eta. They are related to his later notation by: $\Theta(u,q)=\theta_4\big(u\cdot\theta_3(0,q)^{-2},q\big)$, $\Theta_1(u,q)=\theta_3\big(u\cdot\theta_3(0,q)^{-2},q\big)$, $\mathrm{H}(u,q)=\theta_1\big(u\cdot\theta_3(0,q)^{-2},q\big)$ and $\mathrm{H}_1(u,q)=\theta_2\big(u\cdot\theta_3(0,q)^{-2},q\big)$.

representation of the Schrödinger representation. When viewed as a function in the Ω-variable, the theta functions become matrix coefficients of the oscillator representation. This section provides a synopsis of this group-theoretic approach. More systematic treatments can be found in [11, §§1–4 &§8] [2, Ch. 1], as well as the papers [18] and [1].

(2.1) The Heisenberg group. Heisenberg groups are central extensions of abelian groups. We will use the following version of real Heisenberg groups $\mathrm{Heis}(2g,\mathbb{R})$, whose underlying set is $\mathbb{C}_1^\times \times \mathbb{R}^{2g}$, and a typical element will be written as a pair (λ,x), where $\lambda \in \mathbb{C}_1^\times$, $x = \binom{x_1}{x_2}$, $x_1,x_2 \in \mathbb{R}^g$; often we write λ,x_1,x_2 for λ,x. The group law on $\mathrm{Heis}(2g,\mathbb{R})$ is

$$(\lambda,x)\cdot(\mu,y) = \lambda\cdot\mu\cdot\mathbf{e}(\tfrac{1}{2}({}^t x_1\cdot y_2 - {}^t x_2\cdot y_1))$$

so that the $(\lambda,0)$'s form the center of $\mathrm{Heis}(2g,\mathbb{R})$, identified with $\mathbb{C}_1^\times$. This group law induces a $\mathbb{C}_1^\times$-valued commutator pairing

$$\mathbb{R}^{2g}\times\mathbb{R}^{2g} \longrightarrow \mathbb{C}_1^\times, \quad (x,y) \mapsto (\lambda,x)\cdot(\mu,y)\cdot(\lambda,x)^{-1}\cdot(\mu,y)^{-1} = \mathbf{e}({}^t x_1\cdot y_2 - {}^t x_2\cdot y_1)$$

on $\mathbb{R}^{2g}$. Underlying the notation for $\mathrm{Heis}(2g,\mathbb{R})$ is a continuous section $\mathbf{s}$ of the projection $\mathrm{pr}: \mathrm{Heis}(2g,\mathbb{R}) \twoheadrightarrow \mathbb{R}^{2g}$, $\mathbf{s}: x \mapsto (1,x)$ for all $x\in\mathbb{R}^{2g}$, which is a group homomorphism when restricted to any Lagrangian subspace of $\mathbb{R}^{2g}$ for the $\mathbb{R}$-bilinear alternating pairing $E:(x,y)\mapsto {}^t x_1\cdot y_2 - {}^t x_2\cdot y_1$.

(2.2) The Schrödinger representation. The main theorem about the representations of the Heisenberg group $\mathrm{Heis}(2g,\mathbb{R})$, due to Stone, Von Neumann and Mackey is this:

> *There is a unique irreducible unitary representation* $U:\mathrm{Heis}(2g,\mathbb{R}^{2g})\to\mathrm{Aut}(\mathcal{H})$ *whose restriction to the center* $\mathbb{C}_1^\times$ *of* $\mathrm{Heis}(2g,\mathbb{R}^{2g})$ *is* $\lambda\mapsto\lambda\cdot\mathrm{Id}_{\mathcal{H}}$.

The Schrödinger representation can be produced in many ways. For every closed maximal isotropic subgroup $K\subset\mathbb{R}^{2g}$ and every splitting $\sigma: K\to\mathrm{Heis}(2g,\mathbb{R})$ of pr over K, we have the induced representation on the space $L^2(\mathbb{R}^{2g}//K)$ on the space of all essentially L^2-functions on $\sigma(K)\backslash\mathrm{Heis}(2g,\mathbb{R})$ whose restriction to $\mathbb{C}_1^\times$ is $\lambda\mapsto\lambda\cdot\mathrm{Id}$. In the case when $K = 0\oplus\mathbb{R}^g\subset\mathbb{R}^g\oplus\mathbb{R}^g$ and $\sigma = \mathbf{s}|_K$, we have the usual position-momentum realization on the Hilbert space $\mathcal{H}_{\mathrm{PM}} = L^2(\mathbb{R}^{2g})$, where the action of a typical element $(\lambda,y_1,y_2)\in\mathrm{Heis}(2g,\mathbb{R})$ is given by

$$(U_{(\lambda,y_1,y_2)}\phi)(x) = \lambda\cdot\mathbf{e}({}^t x\cdot y_2)\cdot\mathbf{e}(\tfrac{1}{2}{}^t y_1\cdot y_2)\cdot\phi(x+y_1) \quad \forall\phi(x)\in L^2(\mathbb{R}^g).$$

Let $C, A_1,\ldots,A_g, B_1,\ldots,B_g$ be the basis of the Lie algebra $\mathfrak{heis}(2g,\mathbb{R})$ of $\mathrm{Heis}(2g,\mathbb{R})$ such that $\exp(t\cdot C) = (\mathbf{e}(t),\vec{0},\vec{0})$, $\exp(t\cdot A_i) = (1,te_i,\vec{0})$ and $\exp(t\cdot B_i) = (1,\vec{0},te_i)$ for all $i=1,\ldots,g$, where $\vec{e}_1,\ldots,\vec{e}_g$ are the standard basis elements of $\mathbb{R}^g$. Their Lie brackets are

$$[A_i,A_j] = [B_i,B_j] = [C,A_i] = [C,B_j] = 0 \text{ and } [A_i,B_j] = \delta_{ij}\cdot C \quad \forall i,j=1,\ldots,g.$$

The action of $\mathfrak{heis}(2g,\mathbb{R})$ on $\mathcal{H}_{\mathrm{PM}}$ is given by

$$U_{A_i}(f) = \frac{\partial f}{\partial x_i}, \quad U_{B_i}(f) = 2\pi\sqrt{-1}\,x_i\cdot f, \quad U_C(f) = 2\pi\sqrt{-1}\cdot f.$$

(2.3) $\mathcal{H}_\infty$ **and** $\mathcal{H}_{-\infty}$. Let $\mathcal{H}_\infty$ be the set of all smooth vectors in the Schrödinger representation $\mathcal{H}$, consisting of all elements $f \in \mathcal{H}$ such that all higher derivatives

$$\big(\big(\prod_{i=1}^{g} U_{B_i}^{n_i}\big)\circ\big(\prod_{i=1}^{g} U_{A_i}^{m_i}\big)\big)(f)$$

of f exists in $\mathcal{H}$, for all $m_1,\ldots,m_g,n_1,\ldots,n_g \in \mathbb{N}$. By Sobolev's lemma, the space $\mathcal{H}_\infty^{\mathrm{PM}}$ of all smooth vectors in the position-momentum realization $\mathcal{H}_{\mathrm{PM}}$ is equal to the Schwartz space $\mathscr{S}(\mathbb{R}^g)$ of $\mathbb{R}^g$, consisting of all rapidly decreasing smooth functions on $\mathbb{R}^g$.

Let $\mathcal{H}_{-\infty}$ be the space of all $\mathbb{C}$-linear functionals on $\mathcal{H}_\infty$ which are continuous for the topology defined by the family of seminorms

$$\big\{ f \mapsto \big\|\big(\big(\textstyle\prod_{i=1}^{g} U_{A_i}^{m_i}\big)\circ\big(\prod_{i=1}^{g} U_{B_i}^{n_i}\big) f_\Omega\big\| \big\}_{m_i,n_i\in\mathbb{N}\ \forall i=1,\ldots,g}$$

on $\mathcal{H}_\infty$. For the position-momentum realization, this "distribution completion" of $\mathcal{H}$ is the space of all tempered distributions on $\mathbb{R}^g$.

(2.4) The smooth vectors f_Ω. Every element $\Omega \in \mathfrak{H}_g$ determines a Lie subalgebra W_Ω of the complexification $\mathfrak{heis}(2g,\mathbb{C}) := \mathfrak{heis}(2g,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathfrak{heis}(2g,\mathbb{R})$, given by

$$W_\Omega := \textstyle\sum_{i=1}^{g}\mathbb{C}\cdot(A_i - \sum_{j=1}^{g}\Omega_{ij}B_j).$$

A direct computation shows that for every $\Omega \in \mathfrak{H}_g$, the subspace of all elements in the Schrödinger representation killed by W_Ω is a one-dimensional subspace of $\mathcal{H}_\infty$; let f_Ω be a generator of this subspace. In the position-momentum realization, f_Ω can be taken to be the function

$$f_\Omega(x) = \mathbf{e}(\tfrac{1}{2}\,{}^t x\cdot\Omega\cdot x).$$

(2.5) The theta distribution $e_{\mathbb{Z}} \in \mathcal{H}_{-\infty}$. Inside $\mathrm{Heis}(2g,\mathbb{R})$ we have a discrete closed subgroup

$$\sigma(\mathbb{Z}^{2g}) := \big\{\big((-1)^{{}^t m\cdot n}, m, n\big) \in \mathrm{Heis}(2g,\mathbb{R}) \;\big|\; m,n\in\mathbb{Z}^g\big\}.$$

One sees by a direct computation that the set of all elements in $\mathcal{H}_{-\infty}$ fixed by $\sigma(\mathbb{Z}^{2g})$ is a one-dimensional vector subspace over $\mathbb{C}$. A generator $e_{\mathbb{Z}}$ of this subspace in $\mathcal{H}_{\mathrm{PM}}$ is

$$e_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}^g}\delta_n\,, \quad \text{where } \delta_n = \text{the delta function at } n.$$

This element $e_{\mathbb{Z}} \in \mathcal{H}_{-\infty}$ might be thought of as a "universal theta distribution".

(2.6) PROPOSITION. *For any $\Omega \in \mathfrak{H}_g$, any generator f_Ω of $\mathcal{H}^{W_\Omega}$ and any generator $e_{\mathbb{Z}}$ of $\mathcal{H}^{\sigma(\mathbb{Z}^{2g})}$, there exists a constant $c \in \mathbb{C}^\times$ such that*

$$\begin{aligned}\langle U_{(1,x_1,x_2)} f_\Omega,\, e_{\mathbb{Z}}\rangle &= c\cdot\mathbf{e}\big(\tfrac{1}{2}\,({}^t x_1\Omega x_1 + {}^t x_1\cdot x_2)\big)\cdot\theta(\Omega x_1 + x_2;\Omega)\\ &= c\cdot\mathbf{e}(-\tfrac{1}{2}\,{}^t x_1\cdot x_2)\cdot\theta\begin{bmatrix}x_1\\ x_2\end{bmatrix}(0;\Omega)\end{aligned}$$

for all elements $(1,x_1,x_2) \in \mathrm{Heis}(2g,\mathbb{R})$. This constant c is 1 if f_Ω and $e_{\mathbb{Z}}$ are those specified in 2.4 and 2.5 for the position-momentum realization.

(2.7) The big metaplectic group. Let $\widetilde{\mathrm{Mp}}(2g,\mathbb{R})$ be the subgroup of of the group $\mathrm{U}(\mathcal{H})$ of all unitary automorphisms of the Hilbert space $\mathcal{H}$, defined by

$$\widetilde{\mathrm{Mp}}(2g,\mathbb{R}) := \left\{T \in \mathrm{U}(\mathcal{H}) \,|\, \exists\, \gamma \in \mathrm{Sp}(2g,\mathbb{R}) \text{ s.t. } T\cdot U_{(\lambda,x)}\cdot T^{-1} = U_{(\lambda,\gamma x)} \;\; \forall\, (\lambda,x) \in \mathrm{Heis}(2g,\mathbb{R})\right\}$$

The uniqueness of the Schrödinger representation implies that the natural homomorphism

$$\rho : \widetilde{\mathrm{Mp}}(2g,\mathbb{R}) \longrightarrow \mathrm{Sp}(2g,\mathbb{R})$$

is surjective and $\mathrm{Ker}\rho = \mathbb{C}_1^\times \cdot \mathrm{Id}$, so that $\widetilde{\mathrm{Mp}}$ is a central extension of $\mathrm{Sp}(2g,\mathbb{R})$ by $\mathbb{C}_1^\times$. Being a subgroup of $\mathrm{U}(\mathcal{H})$, we have a tautological unitary representation $\widetilde{\mathrm{Mp}}(2g,\mathbb{R})$ on $\mathcal{H}$.

The natural action of the symplectic group $\mathrm{Sp}(2g,\mathbb{R})$ on the set of all Lagrangian subspaces of $\mathbb{R}^{2g}$ induces an action on the set of Lie subalgebras $W_\Omega \subset \mathfrak{heis}(2g,\mathbb{C})$ defined in 2.4. Explicitly, we have $\gamma\cdot W_\Omega = W_{\gamma\cdot\Omega}$, where

$$\gamma\cdot\Omega = (D\cdot\Omega - C)\cdot(-B\cdot\Omega + A)^{-1} \qquad \forall\, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{R}),$$

which is the usual formula after the involution

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto {}^t\gamma^{-1} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}\cdot\gamma\cdot\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}.$$

Let $j(\gamma,\Omega) := \det(-B\cdot\Omega + A)$ for γ as above and $\Omega \in \mathfrak{H}_g$; it is a 1-cocycle for $(\mathrm{Sp}(2g,\mathbb{R}),\mathfrak{H}_g)$.

(2.8) PROPOSITION. *For every $\Omega \in \mathfrak{H}_g$, let f_Ω be the element $f_\Omega(x) = \mathbf{e}(\frac{1}{2}\,{}^t x\Omega x)$ in $\mathcal{H}_{\mathrm{PM}}^{W_\Omega}$.*

(1) *For every $\tilde\gamma \in \widetilde{\mathrm{Mp}}(2g,\mathbb{R})$ with image $\gamma = \rho(\tilde\gamma)$ in $\mathrm{Sp}(2g,\mathbb{R})$ and every $\Omega \in \mathfrak{H}_g$, there exists a unique element $C(\tilde\gamma,\Omega) \in \mathbb{C}^\times$ such that*

$$\tilde\gamma\cdot f_\Omega = C(\tilde\gamma,\Omega)\cdot f_{\gamma\cdot\Omega}.$$

(2) *There is a unitary character $\chi : \widetilde{\mathrm{Mp}}(2g,\mathbb{R}) \to \mathbb{C}_1^\times$ of $\mathrm{Mp}(2g,\mathbb{R})$ such that*

$$C(\tilde\gamma,\Omega)^2 = \chi(\tilde\gamma)\cdot j(\gamma,\Omega) \quad \forall\, \Omega \in \mathfrak{H}_g.$$

(2.9) The metaplectic group and the theta level subgroup. The metaplectic group, defined by $\mathrm{Mp}(2g,\mathbb{R}) := \mathrm{Ker}(\chi)$, is a double cover of the symplectic group $\mathrm{Sp}(2g,\mathbb{R})$. The tautological action of $\mathrm{Mp}(2g,\mathbb{R})$ on $\mathcal{H}$ is called the *oscillator representation*.

Let $\Gamma_{1,2}$ be the subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ consisting of all $\gamma \in \mathrm{Sp}(2g,\mathbb{Z})$ leaving fixed the function $(m,n) \mapsto (-1)^{{}^t m\cdot n}$ on $\mathbb{Z}^{2g}$. Its inverse image $\widetilde{\Gamma_{1,2}}$ in $\mathrm{Mp}(2g,\mathbb{R})$ is the subgroup consisting of all elements $\tilde\gamma \in \mathrm{Mp}(2g,\mathbb{R})$ such that $\tilde\gamma\cdot\sigma(\mathbb{Z}^{2g})\cdot\tilde\gamma^{-1} = \sigma(\mathbb{Z}^{2g})$.

(2.10) PROPOSITION. *Let μ_8 be the group of all 8th roots of unity. There exists a surjective group homomorphism $\eta : \widetilde{\Gamma_{1,2}} \to \mu_8$ such that $\tilde\gamma\cdot e_{\mathbb{Z}} = \eta(\tilde\gamma)\cdot e_{\mathbb{Z}}$ for all $\tilde\gamma \in \widetilde{\Gamma_{1,2}}$.*

Because $\mathrm{Mp}(2g,\mathbb{R})$ is contained in the normalizer of the image of the Heisenberg group in $\mathrm{U}(\mathcal{H})$ by construction, we have a unitary representation of their semi-direct product $\mathrm{Heis}(2g,\mathbb{R}) \rtimes \mathrm{Mp}(2g,\mathbb{R})$ on $\mathcal{H}$ which combines the Schrödinger and the oscillator representation. Proposition 2.11 below, which is a reformulation of 2.11 (1), says that the Riemann theta function $\theta(z;\Omega)$ is essentially a matrix coefficient for the representation of $\mathrm{Heis}(2g,\mathbb{R}) \rtimes \mathrm{Mp}(2g,\mathbb{R})$ up to some elementary exponential factor.

(2.11) PROPOSITION. *Let $e_{\mathbb{Z}}$ be the $\sigma(\mathbb{Z}^{2g})$-invariant distribution $\sum_{n\in\mathbb{Z}^{2g}} \delta_n$ in $\mathcal{H}^{\mathrm{PM}}_{-\infty}$ and let $f_{\sqrt{-1}\mathrm{I}_g}$ be the smooth vector $f_{\sqrt{-1}\mathrm{I}_g}(x) = \exp(-\pi\, {}^t x x)$ in $\mathcal{H}^{\mathrm{PM}}_{\infty}$ fixed by $W_{\sqrt{-1}\mathrm{I}_g}$. We have*

$$\left\langle U_{(1,x_1,x_2)} \cdot \tilde{\gamma} \cdot f_{\sqrt{-1}},\ e_{\mathbb{Z}} \right\rangle = C(\tilde{\gamma}, \sqrt{-1}\mathrm{I}_g) \cdot \mathbf{e}(-\tfrac{1}{2}\, {}^t x_1 x_2) \cdot \theta\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right](0, \gamma \cdot \sqrt{-1}\,\mathrm{I}_g)$$

for all $\tilde{\gamma} \in \mathrm{Mp}(2g,\mathbb{R})$ and all $(1,x_1,x_2) \in \mathrm{Heis}(2g,\mathbb{R})$, where γ is the image of $\tilde{\gamma}$ in $\mathrm{Sp}(2g,\mathbb{R})$.

In the above formula $\gamma \cdot \sqrt{-1}\,\mathrm{I}_g = (\sqrt{-1}D - C) \cdot (A - \sqrt{-1}B)^{-1} =: \Omega_\gamma$ if $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and we have $C(\tilde{\gamma}, \sqrt{-1}\mathrm{I}_g)^2 = \det(A - \sqrt{-1}B)$, which determines $C(\tilde{\gamma}, \sqrt{-1}\mathrm{I}_g)$ up to ± 1. Note that

$$\begin{aligned} &C(\tilde{\gamma}, \sqrt{-1}\mathrm{I}_g) \cdot \mathbf{e}(-\tfrac{1}{2}\, {}^t x_1 x_2) \cdot \theta\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right](0, \gamma \cdot \sqrt{-1}\,\mathrm{I}_g) \\ &= C(\tilde{\gamma}, \sqrt{-1}\mathrm{I}_g) \cdot \mathbf{e}(-\tfrac{1}{2}\,({}^t x_1 \cdot \Omega_\gamma \cdot x_2) + {}^t x_1 \cdot x_2) \cdot \theta(\Omega_\gamma x_1 + x_2; \Omega_\gamma), \end{aligned}$$

so up the the elementary factors $C(\tilde{\gamma}, \sqrt{-1}\mathrm{I}_g) \cdot \mathbf{e}(-\frac{1}{2}\,({}^t x_1 \cdot \Omega_\gamma \cdot x_2) + {}^t x_1 \cdot x_2)$, the Riemann theta function is indeed a matrix coefficient of the group $\mathrm{Heis}(2g,\mathbb{R}) \rtimes \mathrm{Mp}(2g,\mathbb{R})$ on the Hilbert space $\mathcal{H}$ for the unitary representation which combines the Schrödinger and the oscillator representation.

(2.12) Propositions 2.8 and 2.10 provide a modern version of the transformation theory of theta functions. They reflect the fact that theta constants $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0;\Omega)$ with rational characteristics are Siegel modular forms of weight $1/2$. The readers may consult Part 2 of [6] and [7, Ch. V] for what this theory looked like at the end of the nineteenth century.

(2.13) So far we have only looked at matrix coefficients attached to the special elements $f_\Omega \otimes e_{\mathbb{Z}}$ in the tensor product $\mathcal{H}_\infty \otimes \mathcal{H}_{-\infty}$. The group $\mathrm{Heis}(2g,\mathbb{R}) \times \mathrm{Heis}(2g,\mathbb{R})$ acts on the tensor product $\mathcal{H}_\infty \otimes \mathcal{H}_{-\infty}$, such that $\mathbb{C}_1^\times$ in the first copy of $\mathrm{Heis}(2g,\mathbb{R})$ acts on through $z \mapsto z \cdot \mathrm{Id}$ and the $\mathbb{C}_1^\times$ in the second copy of $\mathrm{Heis}(2g,\mathbb{R})$ acts through via $z \mapsto z^{-1} \cdot \mathrm{Id}$. Each element of $\mathbf{v} \in \mathcal{H}_\infty \otimes \mathcal{H}_{-\infty}$ gives rise to a function $f_{\mathbf{v}}$ on $\mathbb{R}^{2g}$, defined by $f_{\mathbf{v}}(x) = \varepsilon\big((U_{(1,x)}, 1) \cdot \mathbf{v}\big)$, where $\varepsilon : \mathcal{H}_\infty \otimes \mathcal{H}_{-\infty} \to \mathbb{C}$ is the linear functional corresponding to the pairing between $\mathcal{H}_\infty$ and $\mathcal{H}_{-\infty}$. The action of the first and second copy of $\mathrm{Hein}(2g,\mathbb{R})$ on $\mathbf{v} \in \mathcal{H}_\infty \otimes \mathcal{H}_{-\infty}$ becomes two commuting actions of $\mathrm{Hein}(2g,\mathbb{R})$ on the space of "good functions" on $\mathbb{R}^{2g}$,

$$U^{\mathrm{right}}_{(\lambda,y_1,y_2)} f(x_1,x_2) = \lambda \cdot \mathbf{e}\big(\tfrac{1}{2}\,({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)\big) \cdot f(x_1 + y_1, x_2 + y_2)$$

and

$$U^{\mathrm{left}}_{(\lambda,y_1,y_2)} f(x_1,x_2) = \lambda^{-1} \cdot \mathbf{e}\big(\tfrac{1}{2}\,({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)\big) \cdot f(x_1 - y_1, x_2 - y_2)$$

respectively. This provides a very nice way to organize all theta functions through symmetries arising from the action of the groupo $\mathrm{Hein}(2g,\mathbb{R}) \times \mathrm{Hein}(2g,\mathbb{R})/\{(\lambda,\lambda) | \lambda \in \mathbb{C}_1^\times\}$. Of coruse the

last group is isomorphic to the Heisenberg group $\mathrm{Hein}(4g,\mathbb{R})$, and its action on $\mathcal{H}_{infty} \otimes \mathcal{H}_{-\infty}$ is an incarnation of the Schrödinger representation of this bigger Heisenberg group! This compelling story is elegantly presented in [11, §§1-5]; see also [2, Ch. I] and [1].

(2.14) The Heisenberg groups play an important role in the study of mathematical foundations of quantum mechanics by Weyl in Ch. II §11 and Ch. IV §15 of [19]. Weil recognized that theta series should be interpreted as automorphic forms for the metaplectic group and developed the adelic theory of oscillator representations in [18] for this purpose. This representation-theoretic approach has since become a core part of the theory of automorphic forms and L-functions.

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