Congruences of Néron models for tori and the Artin conductor

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with an appendix

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§1. Introduction

Let K be a complete local field with a perfect residue field. Then every torus T over K has a Néron model $\underline{T}^{\mathrm{NR}}$ over the ring of integers $\mathcal{O}$ of K. The standard reference for Néron models is the book [BLR]. The torus is canonically determined by the lattice $\Lambda = X_*(T)$ and the action of $\Gamma = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ on Λ. Therefore, $\underline{T}^{\mathrm{NR}}$ depends only on (K, Λ).

For large e, the action of Γ on Λ factors through the quotient $\Gamma(\mathrm{Tr}_e K)$ of Γ. Here, $\Gamma(\mathrm{Tr}_e K)$ is the Galois group classifying extensions of K that are "at most e-ramified", and it depends only on $\mathrm{Tr}_e K$ via the theory of Deligne [D], where $\mathrm{Tr}_e K = (\mathcal{O}/\mathfrak{p}^e, \mathfrak{p}/\mathfrak{p}^{e+1}, \epsilon)$ is Deligne's e^{th} truncation ($\mathfrak{p}$ being the prime ideal of $\mathcal{O}$, and ϵ being the canonical map from $\mathfrak{p}/\mathfrak{p}^{e+1}$ to $\mathcal{O}/\mathfrak{p}^e$).

What does $(\mathrm{Tr}_e K, \Lambda)$ tell us about $\underline{T}^{\mathrm{NR}}$? It is natural to expect that $(\mathrm{Tr}_e K, \Lambda)$ determines $\underline{T}^{\mathrm{NR}} \otimes (\mathcal{O}/\mathfrak{p}^N)$ when $e \gg N$. The main result (9.2) of this paper confirms this expectation. Another version (8.5) of our main result is formulated without Deligne's theory, and is valid for local fields that are not necessarily complete.

This result leads to a solution of a question of Gross, which was communicated to us by G. Prasad and motivated this work. The Néron model does not commute with base field extension before the torus splits. Choose a finite Galois extension L/K which splits T. One can measure the change of Néron models when passing from K to L by comparing $(\mathrm{Lie}\,\underline{T}^{\mathrm{NR}}) \otimes \mathcal{O}_L$ with $\mathrm{Lie}\big(\underline{(T \otimes L)}^{\mathrm{NR}}\big)$. The length of the $\mathcal{O}_L$-module

$$\frac{\mathrm{Lie}\big(\underline{(T\otimes L)}^{\mathrm{NR}}\big)}{(\mathrm{Lie}\,\underline{T}^{\mathrm{NR}})\otimes \mathcal{O}_L}$$

divided by the ramification index of L/K is an invariant $c(T)$ of the torus T. The question is to understand this invariant. An argument in Gross and Gan [GG] shows that if it is known that $c(T)$ depends only on the isogeny class of T (i.e. only on $\Lambda \otimes \mathbb{Q}$) for all T over K, then $c(T)$ is equal to one-half of the Artin conductor of the Galois representation $\Lambda \otimes \mathbb{Q}$.

This invariance under isogeny is not difficult to check when the residue field of $\mathcal{O}$ is of characteristic 0. The case when K is a finite extension of $\mathbb{Q}_p$ can be proved using Tate's formula for the Euler-Poincaré characteristic for Galois cohomologies of local fields, and was known to Chai,

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Gross, and Kushnirsky independently. This argument can be extended to handle any local field of characteristic 0 with perfect residue field, see §11.

When K is of equal-characteristic p, the above method doesn't work and the problem is a major obstruction for extending Gross's theory [G] of the motive of a reductive group to the function field setting in the work of Kushnirsky [K]. Using the main result of this paper, together with Deligne-Kazhdan-Krasner's idea that local fields in characteristic p are limits of local fields of characteristic 0 as the absolute ramification index tending to infinity, we solve the problem in this case.

E. de Shalit has used this technique to obtain interesting arithmetic results [dS]. During the writing of this work, we learned that he has found a very nice solution to Gross's question, also based on Deligne-Kazhdan-Krasner's idea. His method deals with the Lie algebras directly and is different from ours. He has kindly written up his solution as an appendix to this paper. Together with the definition of the invariant $c(T)$ and Theorem 11.3, the appendix provides a complete proof that $c(T)$ is equal to one-half of the Artin conductor of $\Lambda \otimes \mathbb{Q}$.

We now give some indications about the proof of the main result. For simplicity, assume that $\mathcal{O}$ is strictly henselian for the rest of this introduction. Recall that the Néron model is constructed by starting with a suitable model $\underline{T}^0$, and then performing a series of dilatations to $\underline{T}^0$. When one tries to understand the reduction of the Néron model from this construction, several difficulties arise: (i) it is not clear whether $\underline{T}^0 \otimes (\mathcal{O}/\mathfrak{p}^N)$ depends only on the truncated data $(\mathrm{Tr}_e\, K, \Lambda)$; (ii) in general it is difficult to relate the reduction of a model before and after doing a dilatation, unless the model satisfies some strong regularity condition, e.g. smoothness; (iii) the center of (the first) dilatation is the image of $\underline{T}^0(\mathcal{O}) \to \underline{T}^0(\mathcal{O}/\mathfrak{p})$, which is not a priori determined by the truncated data only.

(iii) is resolved by using a result of Elkik, which tells us that the image of $\underline{T}(\mathcal{O}) \to \underline{T}(\mathcal{O}/\mathfrak{p})$ is the same as the image of $\underline{T}(\mathcal{O}/\mathfrak{p}^N) \to \underline{T}(\mathcal{O}/\mathfrak{p})$ under suitable conditions. This type of result also allows us to handle (i) as follows: we find a (non-flat) model $\underline{T}' \supset \underline{T}^0$ whose reduction depends only on the truncated data, then we show that $\underline{T}^0 \otimes (\mathcal{O}/\mathfrak{p}^N)$ can be recovered inside $\underline{T}' \otimes (\mathcal{O}/\mathfrak{p}^N)$ as the schematic closure of a suitable collection of points coming from reduction modulo π^N, which can be controlled by Elkik's result again.

We overcome (ii) by embedding $\underline{T}^0$ into a smooth model $\underline{R}$ of a larger torus, and perform the dilatations to $\underline{R}$ instead. At the end, we need to recover $\underline{T}^{\mathrm{NR}}$ as the schematic closure of T in the dilatation of $\underline{R}$. This is similar to the situation of recovering $\underline{T}^0$ inside $\underline{T}'$, and is handled by an analogous method. There is another way to deal with (ii) by using Elkik's result, but our present exposition may be psychologically neater.

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§2. Notations

(2.1) Let $\mathcal{O} = \mathcal{O}_K$ be a discrete valuation ring with residue field κ and let K be its field of fractions. Let $\pi = \pi_K$ be a prime element of $\mathcal{O}$. The strict henselization and the completion of $\mathcal{O}$ are denoted by $\mathcal{O}^{\mathrm{sh}}$ and $\widehat{\mathcal{O}}$ respectively. Their fields of fractions are denoted by K^{sh} and $\widehat{K}$ respectively. The residue field of $\mathcal{O}^{\mathrm{sh}}$ is denoted by κ^{sep}, as it is a separable closure of κ.

(2.2) Let T be a torus over K, and L/K be a Galois extension such that T is split over L. Put $\Gamma = \mathrm{Gal}(L/K)$ and $\Lambda = X_*(T)$, the cocharacter group of T. Then T is determined by the Γ-module Λ up to a canonical isomorphism.

(2.3) We will occasionally work with another discrete valuation ring $\mathcal{O}_0$. We denote analogous constructs by the same notations with a subscript 0. We introduce a series of congruence notations:

- $(\mathcal{O}, \mathcal{O}_L) \equiv_\alpha (\mathcal{O}_0, \mathcal{O}_{L_0})$ (level N): this means that α is an isomorphism from $\mathcal{O}_L/\pi^N\mathcal{O}_L$ to $\mathcal{O}_{L_0}/\pi_0^N\mathcal{O}_{L_0}$ and induces an isomorphism $\mathcal{O}/\pi^N\mathcal{O} \to \mathcal{O}_0/\pi_0^N\mathcal{O}_0$.
- $(\mathcal{O}, \mathcal{O}_L, \Gamma) \equiv_{\alpha,\beta} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0)$ (level N): this means $(\mathcal{O}, \mathcal{O}_L) \equiv_\alpha (\mathcal{O}_0, \mathcal{O}_{L_0})$ (level N), β is an isomorphism $\Gamma \to \Gamma_0$, and α is Γ-equivariant relative to β: $\alpha(\gamma.x) = \beta(\gamma).\alpha(x)$.
- $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv_{\alpha,\beta,\phi} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)$ (level N): this means that $(\mathcal{O}, \mathcal{O}_L, \Gamma) \equiv_{\alpha,\beta} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0)$ (level N), and ϕ is isomorphism $\Lambda \to \Lambda_0$ which is Γ-equivariant relative to β.
- If it is not necessary to name the isomorphisms (α, β, etc.), we omit them from the notation.

(2.4) In this paper, "X is determined by $(\mathcal{O}/\pi^N\mathcal{O}, \mathcal{O}_L/\pi^N\mathcal{O}_L, \Gamma, \Lambda)$" means: if

$$(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv_{\alpha,\beta,\phi} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0) \;\; (\text{level } N),$$

then there is a canonical isomorphism $X \to X_0$ determined by (α, β, ϕ) (when X is a scheme), or there is a canonical isomorphism determined by (α, β, ϕ) sending X to X_0 (when X is a function, or a point, etc).

(2.5) All rings in this paper are $\mathcal{O}$-algebras or $\mathcal{O}_0$-algebras. If R is an $\mathcal{O}$-algebra, we denote the ring $R/\pi^N R$ by R^N. We do the same for $\mathcal{O}_0$-algebras.

§3. The construction of the Néron models of a torus

(3.1) Néron models We have three notions of Néron models of T.

- The lft Néron model $\underline{T}^{\text{lft NR}}$ as defined in [BLR, Ch. 10], satisfying $\underline{T}^{\text{lft NR}}(\mathcal{O}^{\mathrm{sh}}) = T(K^{\mathrm{sh}})$.
- The smooth model $\underline{T}^{\text{ft NR}}$ with connected generic fiber such that $\underline{T}^{\text{ft NR}}(\mathcal{O}^{\mathrm{sh}})$ is the maximal bounded subgroup of $T(K^{\mathrm{sh}})$. This model is of finite type over $\mathcal{O}$.
- The connected smooth model $\underline{T}^{\text{conn NR}}$ such that $\underline{T}^{\text{conn NR}}(\mathcal{O}^{\mathrm{sh}})$ is the Iwahori subgroup of $T(K^{\mathrm{sh}})$. This model is the neutral component of either of the previous two models. It has connected generic fiber and special fiber, and is of finite type over $\mathcal{O}$.

It is most convenient for us to work with the second notion. So let $\underline{T}^{\text{NR}} = \underline{T}^{\text{ft NR}}$.

(3.2) Construction We now recall the construction of the Néron model of T as explained in [BLR]. Let $\underline{T}^0$ be any group scheme over $\mathcal{O}$ of finite type such that $\underline{T}^0(\mathcal{O}^{\rm sh}) = \underline{T}^{\rm NR}(\mathcal{O}^{\rm sh})$ (recall that $\underline{T}^{\rm NR}(\mathcal{O}^{\rm sh})$ is an intrinsically defined subset of $T(K^{\rm sh})$). Then $\underline{T}^{\rm NR}$ can be obtained by applying the smoothening process to $\underline{T}^0$. The process produces a series of models $\underline{T}^i$, as follows:

Let Z^i be the Zariski closure of $\{x \bmod \pi \in \underline{T}^i(\kappa^{\rm sep}) : x \in \underline{T}^i(\mathcal{O}^{\rm sh})\}$, as a closed subscheme of $\underline{T}^i \otimes \kappa$ with the reduced induced structure. Then $\underline{T}^{i+1}$ is the dilatation of Z^i on $\underline{T}^i$.

Finally, $\underline{T}^{\rm NR} = \underline{T}^i$ for $i \gg 0$. More precisely, if $\delta = \max\{\delta(x) : x \in T^0(\mathcal{O}^{\rm sh})\}$, then $\underline{T}^{\rm NR} = \underline{T}^i$ for all $i \geq \delta$. Here $\delta(x)$ is Néron's measure for the defect of smoothness. See §5 for more discussions.

(3.3) One way to show the existence of $\underline{T}^0$ is to take R to be a torus containing T such that $\underline{R}^{\rm NR}$ is easy to describe. For example, we can take R to be an induced torus (we will do so in a canonical way later). Let $\underline{T}^0$ be the schematic closure of T in $\underline{R}^{\rm NR}$. Then $\underline{T}^0(\mathcal{O}^{\rm sh}) = \underline{T}^{\rm NR}(\mathcal{O}^{\rm sh})$.

There are other ways to achieve this. For example, each $\underline{T}^i$ can serve as "$\underline{T}^0$", because $\underline{T}^i(\mathcal{O}^{\rm sh}) = \underline{T}^{i-1}(\mathcal{O}^{\rm sh})$ by construction. Notice that $\underline{T}^i$ is flat over $\mathcal{O}$ by construction for all $i \geq 1$. But we don't have to start with a flat $\underline{T}^0$.

(3.4) For our purpose, it is convenient to reformulate the above slightly differently. We maintain our assumptions about R and $\underline{R}^{\rm NR}$. But we only assume that $\underline{T}^0$ is a closed subgroup of $\underline{R}^{\rm NR}$ with generic fiber T such that $\underline{T}^0(\mathcal{O}^{\rm sh}) = \underline{T}^{\rm NR}(\mathcal{O}^{\rm sh})$.

We first produce a sequence of smooth models $\underline{R}^i$ of R as follows:

Let $\underline{R}^0 = \underline{R}^{\rm NR}$. For $i \geq 0$, let W^i be the Zariski closure of $\{x \bmod \pi \in \underline{R}^i(\kappa^{\rm sep}) : x \in \underline{T}^0(\mathcal{O}^{\rm sh}) \subset \underline{R}^i(\mathcal{O}^{\rm sh})\}$, as a subscheme of $\underline{R}^i \otimes \kappa$ with the reduced induced structure. Then $\underline{R}^{i+1}$ is the dilatation of W^i on $\underline{R}^i$. Notice that by the defining properties of the dilatation, the inclusion $\underline{T}^0(\mathcal{O}^{\rm sh}) \subset \underline{R}^i(\mathcal{O}^{\rm sh})$ induces an inclusion $\underline{T}^0(\mathcal{O}^{\rm sh}) \subset \underline{R}^{i+1}(\mathcal{O}^{\rm sh})$, which justifies this inductive definition.

(3.5) Lemma *The schematic closure of T in $\underline{R}^i$ is $\underline{T}^i$ for all $i \geq 1$. In particular, it is $\underline{T}^{\rm NR}$ for $i \gg 0$.*

Proof. We proceed by induction. By induction hypothesis, $\underline{T}^{i-1}$ is a closed subgroup of $\underline{R}^{i-1}$, and W^{i-1} is the image of Z^{i-1} in $\underline{T}^{i-1} \to \underline{R}^{i-1}$. It follows from [BLR, 3.2/Prop. 2(cd)] that $\underline{T}^i$ is a closed subgroup of $\underline{R}^i$. It follows that $\underline{T}^i$ is the schematic closure of its generic fiber T in $\underline{R}^i$, because $\underline{T}^i$ is flat. ∎

(3.6) As mentioned earlier, we may and will always choose the torus R canonically: we just take $R = \mathrm{Res}_{L/K}(T \otimes L)$ (Weil's restriction). Notice that $T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ is split and canonically isomorphic to $\Lambda \otimes_{\mathbb{Z}} \mathbb{G}_{\rm m}$. Therefore, $\underline{R}^{\rm NR}$ is simply $\Lambda \otimes_{\mathbb{Z}} \big(\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}}(\mathbb{G}_{{\rm m}/\mathcal{O}_L})\big)$.

Let $\underline{T}^0$ be the schematic closure of T in $\underline{R}^{\rm NR}$ and put $\underline{T}^0_L = \underline{T}^0 \otimes \mathcal{O}_L$, the schematic closure of $T \otimes L$ in $\underline{R}^{\rm NR} \otimes \mathcal{O}_L$. We will also need to consider the following group schemes over $\mathcal{O}_L$: let $\underline{R}' = \underline{R}^{\rm NR} \otimes \mathcal{O}_L$, $\underline{R}^\dagger = X_*(R \otimes_K L) \otimes_{\mathbb{Z}} (\mathbb{G}_{{\rm m}/\mathcal{O}_L})$, $\underline{T}^\dagger = X_*(T \otimes_K L) \otimes_{\mathbb{Z}} (\mathbb{G}_{{\rm m}/\mathcal{O}_L})$. There are canonical morphisms $\underline{T}^\dagger \to \underline{R}^\dagger$, $\varphi : \underline{R}' \to \underline{R}^\dagger$. Let $\underline{T}' = \underline{T}^\dagger \times_{\underline{R}^\dagger} \underline{R}'$. So we have a Cartesian diagram

$$\begin{array}{ccc} \underline{T}' & \longrightarrow & \underline{T}^\dagger \\ \downarrow & & \downarrow \\ \underline{R}' & \xrightarrow{\varphi} & \underline{R}^\dagger \end{array}$$

The vertical arrows are closed immersions. Since $\underline{T}'$ has generic fiber $T \otimes L$, $\underline{T}^0_L$ is equal to the schematic closure of $T \otimes L$ in $\underline{T}'$.

Observe that the whole diagram modulo π^N is determined by $(\mathcal{O}^N, \mathcal{O}^N_L, \Gamma, \Lambda)$. This remark will be useful later in (5.5) and (8.2.2).

(3.7) Remark Since we choose $\underline{T}^0$ canonically, it follows that the model $\underline{T}^i$ is canonically associated to $T_{/K}$ for each i. Our methods also show that the reduction of $\underline{T}^i$ depends only on the truncated data of sufficiently high level.

§4. Dilatation of smooth schemes and reduction

(4.1) As mentioned in the introduction, a difficulty in studying the reduction of $\underline{T}^i$ is the following: if A is the affine ring of $\underline{T}^i$ and Z^i is defined by the ideal $(\pi, f_1, \ldots, f_n)$, then the affine ring of $\underline{T}^{i+1}$ is

$$A[X_1, \ldots, X_n]/(\pi X_1 - f_1, \ldots, \pi X_n - f_n) \bmod \pi^\infty\text{-torsion}.$$

We don't know how to control the above torsion, not even locally.

However, we have the following:

(4.2) Proposition *Let X be a smooth scheme over $\mathcal{O}$, and W be a closed smooth subscheme over $X \otimes \kappa$. Let X' be the dilatation of W on X. Then for any $N \geq 1$, $X' \otimes \mathcal{O}^N$ depends only on $X \otimes \mathcal{O}^{N+1}$ in a canonical way.*

(4.2.1) Canonicity We first explain the canonicity. Recall that X' is an open subscheme of the blow-up $\mathrm{Bl}(X, W)$ of X with center W, which is defined to be

$$\mathrm{Bl}(X, W) = \mathbf{Proj} \bigoplus_{t \geq 0} \mathcal{I}^t,$$

where $\mathcal{I}^t$ is the t^{th} power of the ideal sheaf $\mathcal{I}$ of W in $\mathcal{O}_X$. On the other hand, $\mathrm{Bl}(X, W)$ is a closed subscheme of $\mathrm{Bl}'(X, \mathcal{I})$, where

$$\mathrm{Bl}'(X, \mathcal{I}) = \mathbf{Proj} \bigoplus_{t \geq 0} \mathrm{Sym}^t_{\mathcal{O}_X} \mathcal{I}.$$

The closed immersion $\mathrm{Bl}(X, W) \to \mathrm{Bl}'(X, \mathcal{I})$ is induced by the natural morphism from the symmetric power $\mathrm{Sym}^t \mathcal{I}$ to $\mathcal{I}^t$.

While the dilatation is defined only in a very special context and the blow-up doesn't behave well under base change, $\mathrm{Bl}'(X, \mathcal{I})$ can be defined whenever $\mathcal{I}$ is a coherent sheaf of $\mathcal{O}_X$-module, and its formation clearly commutes with arbitrary base change. It follows that $\mathrm{Bl}'(X, \mathcal{I}) \otimes \mathcal{O}^N = \mathrm{Bl}'(X \otimes \mathcal{O}^N, \mathcal{I} \otimes \mathcal{O}^N)$ depend only on $(X \otimes \mathcal{O}^{N+1}, W)$, because $\pi\mathcal{O}_X \subset \mathcal{I} \subset \mathcal{O}_X$.

Now assume that X_1 and X_2 are $\mathcal{O}$-schemes, and ϕ is an isomorphism $X_1 \otimes \mathcal{O}^{N+1} \to X_2 \otimes \mathcal{O}^{N+1}$. Assume also that $W_1 \subset X_1 \otimes \kappa$, $W_2 \subset X_2 \otimes \kappa$ are closed subschemes smooth over κ and ϕ induces an isomorphism from W_1 to W_2. Form the dilatation X'_i and $Y_i = \mathrm{Bl}'(X_i, \mathcal{I}_i)$, $i = 1, 2$. Our canonicity statement is: the natural isomorphism $\mathrm{Bl}'(\phi) : Y_1 \otimes \mathcal{O}^N \to Y_2 \otimes \mathcal{O}^N$ induces an isomorphism from the subscheme $X'_1 \otimes \mathcal{O}^N$ of $Y_1 \otimes \mathcal{O}^N$ to $X'_2 \otimes \mathcal{O}^N$.

(4.2.2) Proof of Prop. (4.2) The following local calculation is taken from [BLR, 3.2/Prop. 3]. Let $i = 1, 2$. Let x_i' be a point on $X_i' \otimes \kappa$ which projects to $x_i \in X_i \otimes \kappa$. The local calculation there shows that we can choose a system of local coordinates $f_1^{(i)}, \ldots, f_r^{(i)}, g_{r+1}^{(i)}, \ldots, g_n^{(i)}$ at x_i on $X_{i/\mathcal{O}}$ such that W^i is defined by $(\pi, g_{r+1}^{(i)}, \ldots, g_n^{(i)})$ near an affine neighborhood U_i of x_i, X_i' above U_i is simply

$$\operatorname{Spec}(B_i'/\pi^\infty\text{-torsion}), \text{ where } B_i' = \mathcal{O}_{X_i}(U_i)[Y_{r+1}^{(i)}, \ldots, Y_n^{(i)}]/(\pi Y_{r+1}^{(i)} - g_{r+1}^{(i)}, \ldots, \pi Y_n^{(i)} - g_n^{(i)}),$$

and $f_1^{(i)}, \ldots, f_r^{(i)}, Y_{r+1}^{(i)}, \ldots, Y_n^{(i)}$ form a system of local coordinates at x_i' on $X'_{i/\mathcal{O}}$.

It is seen directly that $(B_i')_{x_i'}$ is regular local ring, hence free of π^∞-torsion. It follows that a suitable localization $(B_i')_h$ is already free of π^∞-torsion. By shrinking U, we may assume that B_i' is free of π^∞ torsion.

If $\phi(x_1) = x_2$, we can arrange to have: $\phi^*(f_j^{(2)} \bmod \pi^N) \equiv f_j^{(1)} \bmod \pi^N$, $\phi^*(g_k^{(2)} \bmod \pi^N) \equiv g_k^{(1)} \bmod \pi^N$, and ϕ induces an isomorphism $\tilde{\phi}^* : \mathcal{O}_{X_2}(U_2) \otimes \mathcal{O}^N \to \mathcal{O}_{X_1}(U_1) \otimes \mathcal{O}^N$. Then it is clear that we have an isomorphism $(\phi')^* : B_2' \otimes \mathcal{O}^N \to B_1' \otimes \mathcal{O}^N$ which extends $(\tilde{\phi})^*$ and sends $Y_k^{(2)}$ to $Y_k^{(1)}$.

It remains to show that ϕ' is indeed induced by $\mathrm{Bl}'(\phi)$. Above $U_i \otimes \mathcal{O}^N$, the affine ring of $\mathrm{Bl}'(X_i, \mathcal{I}) \otimes \mathcal{O}^N$ is just

$$B_i'' = \Big(\bigoplus_{t \geq 0} \mathrm{Sym}^t_{B_i^N} I_i^N\Big)^{\deg 0}_{\pi_1}$$

where $B_i^N = \mathcal{O}_{X_i}(U_i) \otimes \mathcal{O}^N$, $I_i^N = (\pi, g_{r+1}^{(i)}, \ldots, g_n^{(i)}) \otimes \mathcal{O}^N$, the subscript π_1 indicates localizing regarding π as a homogeneous element of degree 1, and the superscript $\deg 0$ indicates taking the degree 0 part of the localized ring.

The ring B_i'' maps to $(B_i')^N$ by sending $\pi_1^{-1} g_k^{(i)}$ to Y_k. Now it is clear that $(\phi')^*$ is indeed induced by $\mathrm{Bl}'(\phi)^* : B_2'' \to B_1''$. The lemma is proved. ∎

(4.3) Corollary *For $i \geq 0$, $N \geq 1$, $\underline{R}^{i+1} \otimes \mathcal{O}^N$ depends only on $\underline{R}^i \otimes \mathcal{O}^{N+1}$ in a canonical way.*

Proof. Because W^i is a reduced algebraic group over κ, it is smooth over κ. Therefore, we can apply the proposition. ∎

§5. Bounding the defect of smoothness

(5.1) Change of base field Let X be any scheme of finite type over $\mathcal{O}$ such that $X \otimes K$ is smooth over K. Consider $x \in X(\mathcal{O}^{\mathrm{sh}})$ as a morphism $\operatorname{Spec} \mathcal{O}^{\mathrm{sh}} \to X$. Recall that Néron's measure of the defect of smoothness $\delta(x)$ is defined by

$$\delta(x) = \delta(x; X) = \mathrm{length}_{\mathcal{O}^{\mathrm{sh}}} \mathrm{torsion}(x^* \Omega^1_{X/\mathcal{O}}),$$

the length of the torsion part of $x^*\Omega^1_{X/\mathcal{O}}$. If we consider x as a point of $X \otimes \mathcal{O}_L$, then $\delta(x; X \otimes \mathcal{O}_L) = e(L/K) \cdot \delta(x; X)$, where $e(L/K)$ is ramification index of L/K. This follows from the elementary fact that the formation of Ω^1 commutes with base change.

(5.2) Closed immersion Let $i : X \subset X'$ be a closed immersion of $\mathcal{O}$-schemes such that i induces an isomorphisms $X \otimes K \to X' \otimes K$. Then we have a surjection $i^*\Omega^1_{X'/\mathcal{O}} \to \Omega^1_{X/\mathcal{O}}$. Therefore, for any $x \in X(\mathcal{O}^{\mathrm{sh}})$, we have a surjection $(i \circ x)^*\Omega^1_{X'/\mathcal{O}} \to x^*\Omega^1_{X/\mathcal{O}}$. It follows that $\delta(x; X) \le \delta(i \circ x; X')$.

(5.3) Group scheme Suppose that X is a group scheme over $\mathcal{O}$ and $e \in X(\mathcal{O}^{\mathrm{sh}})$ is the identity element. For each $x \in X(\mathcal{O}^{\mathrm{sh}})$, let $r_x : X \otimes \mathcal{O}^{\mathrm{sh}} \to X \otimes \mathcal{O}^{\mathrm{sh}}$ be the isomorphism of right multiplication by x. Then $r_x \circ e = x$ and hence $e^*\Omega^1_{X/\mathcal{O}^{\mathrm{sh}}} = x^*\Omega^1_{X/\mathcal{O}^{\mathrm{sh}}}$. Therefore, $\delta(e) = \delta(x)$. We conclude that in (3.2), it suffices to take $i \ge \delta(e)$.

(5.4) We would like to control $\delta(e; \underline{T}^0)$. By (5.1), (5.2), and (3.6), we have

$$\delta(e, \underline{T}^0) \le \frac{\delta(e, \underline{T}')}{e(L/K)}.$$

So it is enough to control $\delta(e; \underline{T}')$. According to [BLR, 3.3], $\delta(e; \underline{T}')$ can be calculated by embedding $\underline{T}'$ into a smooth scheme over $\mathcal{O}_L$ as a closed subscheme. We use the embedding $\underline{T}' \to \underline{R}'$.

The schemes $\underline{T}^\dagger$ and $\underline{R}^\dagger$ can be written very explicitly. In fact $\underline{T}^\dagger$ is cut out by $nd - d$ explicit equations $f_1, \dots, f_{nd-d}$ on $\underline{R}^\dagger$, where $d = \dim T$, $n = [L : K]$. It follows that $\underline{T}'$ is cut out by the equations $\varphi^* f_1, \dots, \varphi^* f_{nd-d}$ on $\underline{R}'$.

Now let $z_1, \dots, z_{nd}$ be a system of local coordinates near e, and put

$$M = \Big(\frac{\partial(\varphi^* f_i)}{\partial z_j}\Big)_{\substack{1 \le i \le nd-d \\ 1 \le j \le nd}}.$$

Then $\delta(e; \underline{T}')$ is the minimum of the valuation of $e^*\Delta$, for all $(nd - d)$-minors Δ of M.

(5.5) Lemma *Suppose that* $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)$ (level N) *with* $Ne(L/K) > \delta(e; \underline{T}')$. *Form* $\underline{T}'_0$ *in the same way that we form* $\underline{T}'$. *Then* $\delta(e; \underline{T}'_0) = \delta(e; \underline{T}')$

Proof. For any $N \ge 1$, all the following objects are clearly determined by $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$ only (see the end of (3.6)): $\underline{R}^\dagger \otimes \mathcal{O}_L^N$, $\underline{T}^\dagger \otimes \mathcal{O}_L^N$, $\underline{R}' \otimes \mathcal{O}_L^N$, $\underline{T}' \otimes \mathcal{O}_L^N$, the matrix $e^*(M \bmod \pi^N)$, $\min(\delta(e; \underline{T}'), Ne(L/K))$. Now the lemma is obvious. ∎

§6. Singularities of commutative group schemes

In this section, we assemble some known facts about relative complete intersection (r.c.i.) morphisms to be used later. The basic reference is [EGA IV, 19.3].

(6.1) Lemma *Let G be a commutative group scheme, flat and of finite type over a noetherian base scheme S. Then $G \to S$ is an r.c.i. morphism.*

Proof. By definition [EGA IV, 19.3.6], it suffices to check the lemma when $S = \operatorname{Spec} k$, where k is a field. By [EGA IV, 19.3.9(ii)], the property of being an r.c.i. morphism can be checked after an fpqc (faithfully flat and quasi-compact) base change. Therefore, we may and do assume that k is algebraically closed.

Now suppose that

$$0 \to G' \to G \to G'' \to 0$$

is an exact sequence of commutative group schemes over k. Assume that G' and G'' are r.c.i. over $\operatorname{Spec} k$. We claim that $G \to G''$ is an r.c.i. morphism. Indeed, it suffices to check it after the fpqc base change $G \to G''$, namely, we look at $G \times_{G''} G \to G$. This morphism is canonically isomorphic to $G \times_{\operatorname{Spec} k} G' \to G$ (projection to the first factor), and is an r.c.i. morphism since $G' \to \operatorname{Spec} k$ is. Thus the claim is proved. It follows that the composition $G \to G'' \to \operatorname{Spec} k$ is an r.c.i. morphism, by [EGA IV, 19.3.9(iii)].

Consider an arbitrary G over k. Then G admits a composition series in which the factors are smooth, isomorphic to μ_p, or isomorphic to α_p. These factors are visibly r.c.i. over k. By the above observation, the lemma is proved. ∎

(6.2) Lemma *Suppose that X is a noetherian scheme and $X \to \operatorname{Spec} \mathcal{O}$ is a flat r.c.i. morphism. Then for any $N \geq 1$, The collection of points:*

$$\bigcup \{x \bmod \pi^N \in X(C^N) : x \in X(C)\}$$

as C ranges over local $\widehat{\mathcal{O}}$-algebras which are flat, finite, and r.c.i. over $\widehat{\mathcal{O}}$, is schematically dense in $X \otimes \mathcal{O}^N$.

Proof. Since $\{\operatorname{Spec} \widehat{\mathcal{O}}_{X \otimes \mathcal{O}^N, x} : x \in X \otimes \mathcal{O}^N\}$ is obviously schematically dense in $X \otimes \mathcal{O}^N$, we may assume that $X = \operatorname{Spec} A$ is a complete noetherian local ring such that $\pi \in \mathfrak{m}_A$.

Choose a presentation $A = B/I$, where $B = \widehat{\mathcal{O}}[[X_1, \ldots, X_b]]$. Since the fibers of $\operatorname{Spec} B \to \operatorname{Spec} \widehat{\mathcal{O}}$ are obviously regular, [EGA IV, 19.3.7] implies that $\operatorname{Spec} A \to \operatorname{Spec} B$ is a transversally regular immersion. By definition [EGA IV, 19.2.2], this implies that I is generated by a regular sequence $(t_1, \ldots, t_a)$ on B. By [EGA0, 15.1.6, (b) implying (c)], $(t_1, \ldots, t_a) \otimes \kappa$ is a regular sequence on $B \otimes \kappa$.

Extend $(t_1, \ldots, t_a) \otimes \kappa$ to a regular sequence on $B \otimes \kappa$ of length $b = \dim B \otimes \kappa$, and lift the sequence to a sequence $(t_1, \ldots, t_b)$ in B. Put $J_n = (t_1^n, \ldots, t_b^n)$. Then $\bigcap_n J_n \subset \bigcap_n \mathfrak{m}_B^n = 0$. Let $C_n = B/(I + J_n)$ and $\operatorname{Spec} C_n \to X$ be induced by the obvious map $B/I \to B/(I + J_n)$. Then the points $\{\operatorname{Spec} C_n \to X : n \geq 1\}$ are schematically dense in X.

By [EGA0, 15.1.16, (c) implying (b)], $(t_1, \ldots, t_a, t_{a+1}^n, \ldots, t_b^{n+1})$ is a regular sequence in B and $C_n = B/(t_1, \ldots, t_a, t_{a+1}^n, \ldots, t_b^{n+1})$ is flat. Therefore, C_n is a flat r.c.i.-R-algebra. It is clearly of relative dimension 0 over $\widehat{\mathcal{O}}$. It follows that C_n is finite over $\widehat{\mathcal{O}}$.

It is clear from the above that $\{\operatorname{Spec} C_n \otimes \mathcal{O}^N \to X \otimes \mathcal{O}^N : n \geq 1\}$ is schematically dense in $X \otimes \mathcal{O}^N$. ∎

(6.3) Proposition *Let $\underline{G}$ be a commutative noetherian group scheme over $\mathcal{O}$, not necessarily flat. Let $\bar{\underline{G}}$ be the schematic closure of $\underline{G} \otimes K$ in $\underline{G}$. Then $\bar{\underline{G}} \otimes \mathcal{O}^N$ is the schematic closure in $\underline{G} \otimes \mathcal{O}^N$ of the following collection of points*

$$\bigcup \{x \bmod \pi^N \in \underline{G}(C^N) : x \in \underline{G}(C)\}$$

as C ranges over local $\widehat{\mathcal{O}}$-algebras which are flat, finite, and r.c.i. over $\widehat{\mathcal{O}}$.

Proof. Clear from the two preceding lemmas, together with the fact that $\bar{\underline{G}}(C) = \underline{G}(C)$ for any *flat* $\mathcal{O}$-algebra C. ∎

(6.4) Lemma *The collection of $\mathcal{O}^N$-algebras*

$$\{C^N : C \text{ is a local, flat, finite, r.c.i. } \widehat{\mathcal{O}}\text{-algebra}\}$$

is just the collection of all local $\mathcal{O}^N$-algebras which are flat, finite, and r.c.i. over $\mathcal{O}^N$.

Proof. By [EGA IV, 19.3.9(ii)], the property of being r.c.i. is stable under arbitrary base change. We only need to show that any local, flat, finite, r.c.i. $\mathcal{O}^N$-algebra A is of the form C^N for a suitable C.

Choose any presentation $A = B/I$, $B = \mathcal{O}^N[X_1, \dots, X_n]_{\mathfrak{m}}$, $\mathfrak{m} = (\pi, X_1, \dots, X_n)$. Since the only fiber of $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}^N$ is obviously regular, [EGA IV, 19.3.7] implies that $\operatorname{Spec} A \to \operatorname{Spec} B$ is a transversally regular immersion. By definition ([EGA IV, 19.2.2]), this implies that the ideal $I \subset B$ is generated by a regular sequence $(f_1, \dots, f_m)$. We have $m = n$ since A is of relative dimension 0.

Lift f_i to $\tilde{f}_i \in \widehat{\mathcal{O}}[X_1, \dots, X_n]_{\tilde{\mathfrak{m}}}$, where $\tilde{\mathfrak{m}} = (\pi, X_1, \dots, X_n)$. By [EGA 0, 15.1.16, (c) implying (b)],

$$C = \widehat{\mathcal{O}}[X_1, \dots, X_n]_{\tilde{\mathfrak{m}}}/(\tilde{f}_1, \dots, \tilde{f}_n)$$

is flat and $(\tilde{f}_1, \dots, \tilde{f}_n)$ is a regular sequence on $\widehat{\mathcal{O}}[X_1, \dots, X_n]_{\tilde{\mathfrak{m}}}$. Therefore, C is a local, flat, finite, r.c.i. $\widehat{\mathcal{O}}$-algebra, and $A = C^N$ as desired. ∎

§7. Elkik's theory

(7.1) In this section, Let R be a noetherian $\mathcal{O}$-algebra, complete with respect to the π-adic topology. Consider $R[X] = R[X_1, \dots, X_N]$, the polynomial ring in N variables. Let I be an ideal of $R[X]$ and put $B = R[X]/I$, $Y = \operatorname{Spec} B$. We assume that $Y \otimes_{\mathcal{O}} K \to \operatorname{Spec}(R \otimes_{\mathcal{O}} K)$ is smooth of relative dimension s.

The Jacobian ideal J of I is defined to be the ideal of $R[X]$ generated by the s-minors of

$$\left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1 \le i \le s \\ 1 \le j \le N}}$$

for all $f_1, \dots, f_s$ in a generating set of I.

Because of our smoothness assumption, $J \supset \pi^h R[X]$ for some $h \ge 0$ by the Jacobi criterion [BLR, 2.2/Prop. 7]. Fix such an h in the following.

(7.2) Lemma (Elkik) *Suppose that I can be generated by $N - s$ elements. Then for any $n > 2h$, the image of $Y(R) \to Y(R^{n-h})$ is the same as the image of $Y(R^n) \to Y(R^{n-h})$.* ∎

Proof. This is a special case of [El, p.555, Lemma 1]. Elkik uses an ideal H instead of J as above, and H is defined to be $\sum K_{(\alpha)} \Delta_{(\alpha)}$, summing over certain tuples (α). Examining the definitions, we see that if we set $\alpha = (1, \dots, s)$, then $K_{(\alpha)} = 1$, $\Delta_\alpha = J$, and hence $H \supset J$. Therefore the above lemma follows from Elkik's. ∎

(7.3) Lemma *Suppose that R is a local ring, and $Y \to \operatorname{Spec} R$ is a flat r.c.i. morphism. Then for any $n > 2h$, the image of $Y(R) \to Y(R^{n-h})$ is the same as the image of $Y(R^n) \to Y(R^{n-h})$.*

PROOF. Let $y : \operatorname{Spec} R^n \to Y$ be a point of $Y(R^n)$. Let $\mathfrak{m}$ be the closed point of $\operatorname{Spec} R^n$, $\mathfrak{q} = y(\mathfrak{m})$.

Since $Y \to \operatorname{Spec} R$ is r.c.i. at $\mathfrak{q}$ and $\operatorname{Spec} R[X] \to \operatorname{Spec} R$ has regular fibers, f is a transversally regular immersion at $\mathfrak{q}$ ([EGA IV, 19.3.7]). By ([EGA IV, 19.2.2]), there is some $f \in R[X]$ such that $\mathfrak{q} \in Y_f \subset \operatorname{Spec} R[X]_f$ and Y_f is cut out by $N-s$ equations on $\operatorname{Spec} R[X]_f = \operatorname{Spec} R[X_1, \ldots, X_N, Z]/(Zf(X)-1)$. Because R is local, y maps $\operatorname{Spec} R^n$ to Y_f, i.e. $y \in Y_f(R^n)$.

Let $\{(g_1(X,Z), \ldots, g_{N-s}(X,Z)\}$ be the equations definition Y_f on $\operatorname{Spec} R[X]_f$. Then we can regard Y_f as a closed subscheme of $\operatorname{Spec} R[X,Z] = \mathbb{A}_R^{N+1}$ defined by the $N+1-s$ equations

$$\{g_1(X,Z), \ldots, g_{N-s}(X,Z), Zf(X)-1\}.$$

Let $J' \subset R[X,Z]$ be the Jacobian ideal of these equations. Since J' is also generated by I and $Zf(X)-1$, we see that $\pi^h \in J'$.

By the previous lemma, we can find $\tilde{y} \in Y_f(R) \subset Y(R)$ such that $\tilde{y} \equiv y \pmod{\pi^{n-h}}$. This proves the lemma. ∎

§8. The main argument

For simplicity, we assume that K is complete in this section. However, this assumption is not essential. See (8.6) for more comments.

(8.1) The invariant h We will use the group schemes defined in (3.6). Recall from (5.4) that the closed subscheme $\underline{T}'$ of $\underline{R}'$ is cut out by $(\dim R - \dim T)$ equations, and $\underline{R}' = \underline{R} \otimes \mathcal{O}_L$.

Since $\underline{R}$ is the Néron model of an induced torus, we can write down its defining equation explicitly. In fact, we can realize $\underline{R}$ as a closed subscheme of $\mathbb{A}^{d(n+1)}_{/\mathcal{O}}$, defined by n explicit equations.

We conclude that $\underline{T}'$ can be realized as a closed subscheme of $\mathbb{A}^{d(n+1)}_{/\mathcal{O}_L}$ defined by an ideal I' generated by $d(n+1) - \dim T$ equations. Let J' be the Jacobian ideal for I'. Since the generic fiber of $\underline{T}'$ is smooth, J' contains π^h for some $h \geq 0$. Let $h = h(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda)$ be the smallest integer with this property.

Lemma *Suppose* $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)$ (level N). *Form the Jacobian ideals J' and J'_0 and define the integers h and h_0 for both data. If $h < N$ or $h_0 < N$, then $h = h_0$.*

PROOF. Assume $h < N$. The explicit equations generating J' only depends only on $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda)$, and their reductions modulo π^N depends only on $(\mathcal{O}^N, \mathcal{O}_L^N, \mathcal{O}, \mathcal{O}_L)$. The formation of the Jacobian ideal commutes with base change. Therefore, $J' \otimes \mathcal{O}^N$ depends only on $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$. Therefore, our hypothesis implies

$$J'_0 + \pi_0^N \mathcal{O}_L[X_1, \ldots, X_{d(n+1)}] \supset \pi_0^h \mathcal{O}_L[X_1, \ldots, X_{d(n+1)}].$$

Since $J'_0 \supset \pi^M \mathcal{O}_L[X_1, \ldots, X_{d(n+1)}]$ for some $M \gg 0$, we have $J'_0 \supset \pi_0^h \mathcal{O}_L[X_1, \ldots, X_{d(n+1)}]$ by Nakayama's lemma. Therefore, $h_0 \leq h < N$. The same argument shows that $h \leq h_0$. So $h = h_0$. ∎

If $h < N$, we define $h(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$ to be h, otherwise, we define $h(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda) = N$. This definition is justified by the lemma.

(8.2) Proposition *The group scheme $\underline{T}_L^0 \otimes \mathcal{O}_L^{N-h}$ is determined by $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$ if $N > 2h$.*

(8.2.1) Lemma *Let C be a local $\mathcal{O}_L$-algebra which is π-adically complete. Then the image of $\underline{T}'(C) \to \underline{T}'(C^{N-h})$ is the same as the image of $\underline{T}'(C^N) \to \underline{T}'(C^{N-h})$ if $N > 2h$.*

Proof. By the discussion of (8.1), $\underline{T}'$ is defined by $d(n+1) - \dim T$ equations on $\mathbb{A}^{d(n+1)}$, so the result follows from Lemma 7.2. ∎

(8.2.2) We now prove the proposition. First recall (see the end of (3.6)) that $\underline{T}' \otimes \mathcal{O}_L^N$ is determined by $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$.

By (6.3), it suffices to show that the collection of points $\bigcup_C \operatorname{image}\bigl(\underline{T}'(C) \to \underline{T}'(C^{N-h})\bigr)$, where C ranges over all flat, finite, r.c.i. $\widehat{\mathcal{O}}_L$-algebras, is determined by $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$.

By Lemma 8.2.1 and Lemma 6.4, this collection of points is the same as the union of

$$\operatorname{image}\bigl(\underline{T}'(C^N) \to \underline{T}'(C^{N-h})\bigr)$$

over all flat, finite, r.c.i. $\mathcal{O}^N$-algebras C. From this description it is clear that our collection of points is determined by $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$. ∎

(8.2.3) Remark The argument above shows the following: with the notation of (6.3), assume that $\underline{G} \otimes K$ is smooth of dimension s, and $\underline{G}$ can be defined by $n-s$ equations on $\mathbb{A}^n_{/\mathcal{O}}$, then there is an integer $h \geq 0$ such that $\underline{G} \otimes \mathcal{O}^N$ determines $\bar{\underline{G}} \otimes \mathcal{O}^{N-h}$ for all $N > 2h$.

(8.2.4) Corollary *The group scheme $\underline{T}^0 \otimes \mathcal{O}^{N-h}$ is determined by $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$ for $N > 2h$.*

Proof. This follows from the proposition, and the following elementary statement: Suppose X, X' are closed S-subschemes of an S-scheme Y such that $X \times_S S' = X \times_S S'$ in $Y \times_S S'$ for some $S' \to S$ faithfully flat, then $X = X'$. ∎

(8.3) In the following, we apply the procedure of (3.4), where R is as in (3.6).

We know (cf. (8.1)) that $\underline{T}^0$ is a closed subscheme of $\mathbb{A}^{d(n+1)}$, defined by an ideal I. We also know that I modulo π^{N-h} depends only on $(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$ if $N > 2h$.

Let $J \subset \mathcal{O}[X_1, \ldots, X_{d(n+1)}]$ be the Jacobian ideal of I. Since $I' \subset I$, we have $J' \subset J$ and $\pi^h \in J$.

(8.4) Proposition i) *$\underline{T}^0 \otimes \mathcal{O}^N$ is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ for all $N \geq 1$, $m \geq \max(N + h, 2h+1)$.*

ii) *$\underline{R}^i \otimes \mathcal{O}^{m-i}$ depends only on $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda))$ for all $m \geq \max(2h+i, 3h+1)$.*

iii) *W^i depends only on $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ for $m \geq \max(2h+i+1, 3h+1)$.*

(8.4.1) Lemma *If $N > 2h$, then the image of the map $\underline{T}^0(\mathcal{O}^{\mathrm{sh}}) \to \underline{T}^0\bigl((\mathcal{O}^{\mathrm{sh}})^{N-h}\bigr)$ is the same as the image of $\underline{T}^0\bigl((\mathcal{O}^{\mathrm{sh}})^N\bigr) \to \underline{T}^0\bigl((\mathcal{O}^{\mathrm{sh}})^{N-h}\bigr)$.*

Proof. Since $\underline{T}^0$ is a flat group scheme over $\mathcal{O}$, it is r.c.i. over $\mathcal{O}$ by Lemma 6.1. Now we can apply Lemma 7.3. ∎

(8.4.2) We now prove the proposition. (i) is immediate from Corollary 8.2.4: $\underline{T}^0 \otimes \mathcal{O}^N$, is determined by $\underline{T}^0 \otimes \mathcal{O}^{\max(N,h+1)}$, hence is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ for all $m \geq \max(N, h+1) + h$.

We prove (ii) and (iii) by induction on i. First we look at the case $i = 0$. (ii) is clear: $\underline{R}^0(A) = \Lambda \otimes_{\mathbb{Z}} \big((A \otimes \mathcal{O}_L)^\times\big)$ for any $\mathcal{O}$-algebra A, in particular, for any $\mathcal{O}^N$-algebra A (so actually, in case $i = 0$, (ii) holds without any restriction on N). (iii): Clearly, W^0 is determined by the image of $\underline{T}^0(\mathcal{O}^{\mathrm{sh}}) \to \underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^N\big)$ for any $N \geq 1$. In particular, it is determined by the image of $\underline{T}^0(\mathcal{O}^{\mathrm{sh}}) \to \underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{h+1}\big)$. By Lemma 8.4.1, this image is determined by $\underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{2h+1}\big)$, hence is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ for all $m \geq 3h + 1$ by (i).

Now assume that $i > 0$. (ii) follows immediately from Lemma 4.2 and our induction hypothesis.

(iii): Let B^i be the affine ring of R^i. Recall that

$$B^i = B^{i-1}[Y_1, \dots, Y_m]/(\pi Y_1 - g_1, \dots, \pi Y_m - g_m) \bmod \pi^\infty\text{-torsion},$$

where $(\pi, g_1, \dots, g_m)$ is the ideal defining W^{i-1}. We will write suggestively $\pi^{-1}g_i$ for Y_i.

A point y on $\underline{R}^i$ is determined by the projection of y on $\underline{R}^{i-1}$, together with the additional "coordinates" $(\pi^{-1}g_1)(y), \dots, (\pi^{-1}g_m)(y)$.

Now consider $x \in \underline{T}^0(\mathcal{O}^{\mathrm{sh}})$. Regarding x as an $\mathcal{O}^{\mathrm{sh}}$-valued point on $\underline{R}^i$, we denote it by x_i. Clearly, $x_i \bmod \pi$ is determined by $x_{i-1} \bmod \pi^2$. Inductively, the image of $\underline{T}^0(\mathcal{O}^{\mathrm{sh}}) \to \underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{i+1}\big)$ determines W^i. As in the case $i = 0$, we deduce that this image is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ whenever $m \geq \max(2h + i + 1, 3h + 1)$.

The proposition is proved. ∎

(8.5) The main theorem Let

$$\delta = \left\lfloor \frac{\delta(e, \underline{T}')}{e(L/K)} \right\rfloor .$$

Observe that we have $\delta \leq h$. If $\delta < N$, we define $\delta(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda)$ to be δ, otherwise, we define $\delta(\mathcal{O}^N, \mathcal{O}_L^N, \Gamma, \Lambda) = N$. The definition is justified by Lemma (5.5).

Theorem *Suppose that* $N \geq 1$, $m \geq \max(N + \delta + 2h, 3h + 1)$, *where*

$$\begin{cases} h = h(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda) & \text{as defined in (8.1)}, \\ \delta = \delta(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda) & \text{as defined above.} \end{cases}$$

Then $\underline{T}^{\mathrm{NR}} \otimes \mathcal{O}^N$ *is determined by* $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$.

Notice that since $\delta \leq h$, the hypothesis of the theorem is implied by $m \geq N + 3h$.

(8.5.1) Lemma *Let* X *be a smooth scheme over* $\mathcal{O}$. *Then the schematic closure of the points*

$$\big\{x : x \in X((\mathcal{O}^{\mathrm{sh}})^N)\big\} = \big\{x \bmod \pi^N : x \in X(\mathcal{O}^{\mathrm{sh}})\big\}$$

in $X \otimes \mathcal{O}^N$ *is the whole* $X \otimes \mathcal{O}^N$.

Proof. It suffices to show that if $X = \operatorname{Spec} A$ is smooth over $\mathcal{O}^N$, then $X\big((\mathcal{O}^{\mathrm{sh}})^N\big)$ is schematically dense in X.

Suppose that $f \in A$ is such that $x^*f = 0 \in (\mathcal{O}^{\mathrm{sh}})^N$ for all $x \in X\big((\mathcal{O}^{\mathrm{sh}})^N\big)$. Then $f \bmod \pi$ is zero on every closed point of $X \otimes \kappa^{\mathrm{sep}}$. Therefore, $f \in \pi A$. An easy induction shows that $f = 0$. ∎

(8.5.2) We now prove the theorem. By (3.4), (5.4), and the end of (3.2), $\underline{T}^{\mathrm{NR}}$ is the schematic closure of T in $\underline{R}^{\delta}$.

Let Y be the image of $\underline{T}^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^N\big)$ in $\underline{R}^{\delta}\big((\mathcal{O}^{\mathrm{sh}})^N\big)$. Then the schematic closure of Y in $\underline{R}^{\delta}\otimes\mathcal{O}^N$ is simply $\underline{T}^{\mathrm{NR}}\otimes\mathcal{O}^N$ by the preceding lemma. We know that $\underline{R}^{\delta}\otimes\mathcal{O}^N$ is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ by Proposition 8.4(ii). Therefore, it suffices to show that Y is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$.

As explained in the proof of Proposition 8.4(iii), Y is determined by the image of

$$\underline{T}^0(\mathcal{O}^{\mathrm{sh}}) \to \underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{\delta+N}\big),$$

which is determined by the image of $\underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})\big) \to \underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{\max(N+\delta,h+1)}\big)$, which is the same as the image of

$$\underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{\max(N+\delta,h+1)+h}\big) \to \underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{\max(\delta+N,h+1)}\big)$$

by Lemma 8.4.1.

By Corollary 8.2.4, $\underline{T}^0\big((\mathcal{O}^{\mathrm{sh}})^{\max(\delta+N,h+1)+h}\big)$ is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$. The theorem has been proved completely. ∎

(8.6) Remark For the simplicity of exposition, we have assumed that K is complete. This is because we want to apply Elkik's result in Lemma 8.4.1. We can lift this restriction by either using the henselian version of Elkik's result, or by observing that both $\underline{T}^{\mathrm{NR}}\otimes\mathcal{O}^N$ and $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda)$ remain unchanged when we replace K by its completion.

§9. Application of Deligne's theory

(9.1) Deligne's theory We recall some notions and results from Deligne's article [D]. In this section, assume that K is complete and the residue field κ is perfect.

Let $e \geq 1$. A Galois extension L/K is *at most e-ramified* if $\mathrm{Gal}(L/K)^e = \{1\}$, where the superscript e refers to the upper numbering filtration by the ramification groups [S1]. In other words, $\mathrm{Gal}(L/K)$ is a quotient of $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^e$. Deligne [D] shows that

$$\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^e,$$

together with its upper numbering filtration, is canonically determined by $\mathrm{Tr}_e K$, which is defined in the introduction. Therefore, we may denote $\mathrm{Gal}(K^{\mathrm{sep}}/K)/\mathrm{Gal}(K^{\mathrm{sep}}/K)^e$ by $\Gamma(\mathrm{Tr}_e K)$.

Suppose that $\mathrm{Tr}_e K$ is isomorphic to $\mathrm{Tr}_e K_0$ and L/K is a Galois extension, at most e-ramified. Then there is a corresponding L_0/K_0. This can be explicated as follows ([D, 1.3]). Suppose that $\varphi : \mathcal{O}^e \to \mathcal{O}_0^e$, $\eta : \mathfrak{p}/\mathfrak{p}^{e+1} \to \mathfrak{p}_0/\mathfrak{p}_0^{e+1}$ defines the isomorphism $\mathrm{Tr}_e K \to \mathrm{Tr}_e K_0$. Let π_L be a prime element of $\mathcal{O}_L$ satisfying the Eisenstein equation

$$X^n + \sum_{i=1}^{n-1} a^{(i)} X^i = 0$$

with $a^{(i)} \in \mathfrak{p}$. Let $a_0^{(i)} \in \mathcal{O}_0$ be a life of $\eta(a^{(i)} \bmod \mathfrak{p}^{e+1})$. Then the equation $X^n + \sum_{i=1}^{n-1} a_0^{(i)} X^i = 0$ defines the extension L_0/K_0.

It is clear from the above description that the isomorphism $\mathrm{Tr}_e K \to \mathrm{Tr}_e K_0$ determines canonically an isomorphism $\mathcal{O}_L/\pi^e \to \mathcal{O}_{L_0}/\pi_0^e$, which is $\Gamma(\mathrm{Tr}_e K)$-equivariant.

(9.2) Reformulation of the main theorem Now let S be a triple isomorphic to $\mathrm{Tr}_e\, K$, and Λ be a $\mathbb{Z}$-lattice on which $\Gamma(S)$ acts continuously. Let Γ be the (finite) image of $\Gamma(S)$ in $\mathrm{Aut}(\Lambda)$. Let L/K be the extension whose Galois group is Γ. By the preceding discussion, (S, Λ) determines $(\mathcal{O}^e, \mathcal{O}_L^e, \Gamma, \Lambda)$ canonically.

We define $h(S, \Lambda)$ to be $h(\mathcal{O}^e, \mathcal{O}_L^e, \Gamma, \Lambda)$. Now we have the following reformulation of Theorem 8.5:

Theorem *Let $S = (R, M, \epsilon)$ be a triple as in [D, 1.1], and let Λ be a lattice on which $\Gamma(S)$ acts continuously. Suppose that $N \geq 1$, $e \geq N + 3h(S, \Lambda)$. Then there exists canonically a smooth group scheme $\mathcal{T}_N(S, \Lambda)$ defined over $R/\mathfrak{m}_R^N$ with the following property: for any complete local field K and for any isomorphism $(\varphi, \eta) : S \to \mathrm{Tr}_e\, K$, regard Λ as a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module via Deligne's theory and let $T_{/K}$ be the associated torus, then there is a canonical isomorphism*

$$\mathcal{T}_N(S, \Lambda) \otimes_\varphi \mathcal{O}^N \simeq \underline{T}^{\mathrm{NR}} \otimes \mathcal{O}^N. \blacksquare$$

(9.3) Maintain the preceding notations. As in (3.6), we denote the Néron model of $T \otimes L$ by $\underline{T}^\dagger$. We have trivially that the diagram in (3.6) modulo π^m is determined by (S, Λ).

The closed immersion $T_L^0 \otimes \mathcal{O}_L^N \to \underline{T}^\dagger \otimes \mathcal{O}_L^N$ is determined by (S, Λ). This is clear from the proof of (8.2). From our main argument, it is clear that the morphisms $\underline{R}^{i+1} \otimes \mathcal{O}^N \to \underline{R}^i \otimes \mathcal{O}^N$, as well as the closed immersion $\underline{T}^{\mathrm{NR}} \otimes \mathcal{O}^N \to \underline{R}^\delta \otimes \mathcal{O}^N$, are determined by (S, Λ).

It follows that the morphism $\underline{T}^{\mathrm{NR}} \otimes \mathcal{O}^N \to \underline{R}^0 \otimes \mathcal{O}^N$ is determined by (S, Λ). This morphism factors through the closed immersion $\underline{T}^0 \otimes \mathcal{O}^N \to \underline{R}^0 \otimes \mathcal{O}^N$. It follows that $\underline{T}^{\mathrm{NR}} \otimes \mathcal{O}^N \to \underline{T}^0 \otimes \mathcal{O}^N$ is determined by (S, Λ).

Finally, we conclude that the morphism

$$\underline{T}^{\mathrm{NR}} \otimes \mathcal{O}_L^N \to \underline{T}^\dagger \otimes \mathcal{O}_L^N$$

is determined by (S, Λ).

(9.4) The construction $\Lambda \mapsto \mathcal{T}_N(S, \Lambda)$ is functorial: Let Λ_1, Λ_2 be two lattices on which $\Gamma(S)$ acts continuously and let $\lambda : \Lambda_1 \to \Lambda_2$ is a homomorphism of $\Gamma(S)$-modules, then for $e \gg N$ there is a canonical morphism $\mathcal{T}_N(S, \lambda) : \mathcal{T}_N(S, \Lambda_1) \to \mathcal{T}_N(S, \Lambda_2)$ which corresponds to the reduction modulo π^N of the canonical morphism $\lambda_* : \underline{T}_1^{\mathrm{NR}} \to \underline{T}_2^{\mathrm{NR}}$ induced by λ.

We sketch the proof: let L/K be a Galois extension splitting both T_1 and T_2. Let

$$h_i = h(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda_1), \quad i = 1, 2.$$

Assume that $m \geq N + 4\max(h_1, h_2)$.

From the preceding paragraph, we know that the morphism $\underline{T}_i^{\mathrm{NR}} \otimes \mathcal{O}^{N+h} \to \underline{R}_i^{\mathrm{NR}} \otimes \mathcal{O}^{N+h}$ is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda_i)$. Given a point point $x \in \underline{T}_i^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^N\big)$, lift x to $\tilde{x} \in \underline{T}_i^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big)$, then the image of $\tilde{x}$ in $\underline{R}_i^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big)$ determines x by the proof of Prop. 8.4(iii).

Consider the commutative diagram:

$$\begin{tikzcd}
\underline{R}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \arrow[r] & \underline{R}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \\
\underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \arrow[r, "?"] \arrow[u] \arrow[d] & \underline{T}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \arrow[u] \arrow[d] \\
\underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N}\big) \arrow[r, "?"] & \underline{T}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N}\big)
\end{tikzcd}$$

The whole diagram, except possibly the two arrows labelled with question marks, is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda_1, \Lambda_2)$. Given $x \in \underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^N\big)$, we lift x to $\tilde{x} \in \underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big)$. Let y be the image of $\tilde{x}$ under the composition $\underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \to \underline{R}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \to \underline{R}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big)$. Then y is also the image of $\tilde{x}$ under the composition $\underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \to \underline{T}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big) \to \underline{R}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^{N+h}\big)$. It follows that y determines $\lambda_*(x) \in \underline{T}_2^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^N\big)$.

From the schematic density of $\underline{T}_1^{\mathrm{NR}}\big((\mathcal{O}^{\mathrm{sh}})^N\big)$ in $\underline{T}_1^{\mathrm{NR}} \otimes \mathcal{O}^N$, it follows that the morphism

$$\underline{T}_1^{\mathrm{NR}} \otimes \mathcal{O}^N \to \underline{T}_2^{\mathrm{NR}} \otimes \mathcal{O}^N$$

is determined by $(\mathcal{O}^m, \mathcal{O}_L^m, \Gamma, \Lambda_1, \Lambda_2)$.

§10. The invariant $c(T)$

(10.1) As in the introduction, we define an invariant of a torus T over K as follows: by the universal property of the Néron model, there is a canonical morphism

$$\mathrm{can}_{T,L} : \underline{T}^{\mathrm{NR}} \otimes \mathcal{O}_L \to \underline{(T \otimes L)}^{\mathrm{NR}}$$

extending the identity morphism on the generic fibers. Then $\mathrm{can}_{T,L}$ induces an inclusion on the Lie algebras. We define

$$c(T) = \frac{1}{e(L/K)} \operatorname{length}_{\mathcal{O}_L} \frac{\mathrm{Lie}\big(\underline{(T \otimes L)}^{\mathrm{NR}}\big)}{(\mathrm{can}_{T,L})_*(\mathrm{Lie}\, \underline{T}^{\mathrm{NR}} \otimes \mathcal{O}_L)}.$$

It is easy to see that this is independent of the splitting field L.

The module of translation invariant 1-forms on $\underline{T}^{\mathrm{NR}}_{/\mathcal{O}}$ (resp. $T_{/K}$) is a free $\mathcal{O}$-module (resp. K-vector space) of rank $n = \dim T$ (see [BLR, 4.2]), and is canonically isomorphic to $\mathrm{Hom}_{\mathcal{O}}(\mathrm{Lie}\, \underline{T}^{\mathrm{NR}}, \mathcal{O})$ (resp. $\mathrm{Hom}_K(\mathrm{Lie}\, T, K)$). The top exterior power of this module can be identified with the space of translation invariant n-forms on $\underline{T}^{\mathrm{NR}}_{/\mathcal{O}}$ (resp. $T_{/K}$), and is denoted by $\omega(\underline{T}^{\mathrm{NR}})$ (resp. $\omega(T)$).

Then we can also define $c(T)$ by

$$c(T) = \frac{1}{e(L/K)} \operatorname{length}_{\mathcal{O}_L} \frac{\omega(\underline{T}^{\mathrm{NR}}) \otimes \mathcal{O}_L}{\mathrm{can}^*_{T,L}\, \omega\big(\underline{(T \otimes L)}^{\mathrm{NR}}\big)}$$

(10.2) Proposition *The following two assertions are equivalent:*

i) $c(T_1) = c(T_2)$ *for any two tori* T_1, T_2 *over* K *such that* T_1 *is isogenous to* T_2 *over* K.

ii) $c(T) = \frac{1}{2}a(X_*(T)\otimes\mathbb{Q})$ *for any torus* T *over* K*, where* $a(-)$ *is the Artin conductor of a module over* $\mathbb{Q}[\mathrm{Gal}(K^{\mathrm{sep}}/K)]$*.*

Proof. This is essentially in [GG]. We briefly sketch their argument. Clearly, we have (ii) implying (i). By a direct calculation, we can show that $c(T) = \frac{1}{2}a(X_*(T)\otimes\mathbb{Q})$ when $T = \mathrm{Res}_{L/K}\mathbb{G}_{\mathrm{m}}$ is an induced tori.

Assume (i). Then the function $X_*(T) \mapsto c(T)$ extends to an additive function on $R\otimes\mathbb{Q}$ with values in $\mathbb{Q}$, where R is the ring of virtual representations over $\mathbb{Q}$ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. A theorem of Artin [S2, 9.2] implies that $R\otimes\mathbb{Q}$ is generated by $X_*(T)\otimes\mathbb{Q}$ as T runs through tori of the form $\mathrm{Res}_{L/K}\mathbb{G}_{\mathrm{m}}$. Now the proposition is obvious. ∎

(10.3) The rest of this section gives an interesting condition which is equivalent to the assertion that $c(T)$ is invariant under isogeny.

The space $\omega(T\otimes L)$ is canonically isomorphic to $X^*(T)\otimes L$, where $X^*(T)$ is the character group of T.

Now suppose that we have two tori T_1 and T_2 over K and an isogeny $\alpha : T_1 \to T_2$ defined over K corresponding to $\alpha^* : X^*(T_2) \to X^*(T_1)$. Then the isomorphism

$$\alpha^\sharp = \frac{1}{\deg f}\bigwedge^{\mathrm{top}}(\alpha^*) : \bigwedge^{\mathrm{top}} X^*(T_2) \to \bigwedge^{\mathrm{top}} X^*(T_1)$$

induces an isomorphism

$$(\alpha\otimes L)^\sharp : \omega(T_2\otimes L) \to \omega(T_1\otimes L).$$

It is easy to show ([GG]) that $(\alpha\otimes L)^\sharp$ descends to an isomorphism $\alpha^\sharp : \omega(T_2) \to \omega(T_1)$. We call $\alpha^\sharp$ the *modified-pullback*. When K is of characteristic 0, it is simply the ordinary pull-back divided by $\deg\alpha$.

(10.4) Lemma *With above notations.* $c(T_1) = c(T_2)$ *if and only if* $\alpha^\sharp : \omega(T_2) \to \omega(T_1)$ *induces an isomorphism*

$$\alpha^\sharp : \omega(\underline{T}_2^{\mathrm{NR}}) \to \omega(\underline{T}_1^{\mathrm{NR}}).$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
\omega(T_1)\otimes L & \xrightarrow{\mathrm{can}^*_{T_1,L}} & \omega(T_1\otimes L) \\
\Big\uparrow{\scriptstyle \alpha^\sharp\otimes L} & & \Big\uparrow{\scriptstyle (\alpha\otimes L)^\sharp} \\
\omega(T_2)\otimes L & \xrightarrow{\mathrm{can}^*_{T_2,L}} & \omega(T_2\otimes L)
\end{array}$$

in which each arrow is an isomorphism and each object X is a 1-dimensional vector space over L containing a canonical $\mathcal{O}_L$-lattice M_X. For each arrow $g : X \to Y$, we let $c(g) = \mathrm{length}\, M_Y/g(M_X)$ if $M_Y \supset g(M_X)$, $c(g) = -\,\mathrm{length}\, g(M_X)/M_Y$ otherwise. Clearly, $c(g\circ h) = c(g) + c(h)$.

It is clear that $c((\alpha\otimes L)^\sharp) = 0$. Therefore, $c(\alpha^\sharp\otimes L) = 0$ if and only if $c(\mathrm{can}_{T_1,L}) = c(\mathrm{can}_{T_2,L})$. By definition, $c(\alpha^\sharp\otimes L) = 0$ if and only if $\alpha^\sharp(\omega(\underline{T}_2^{\mathrm{NR}})) = \omega(\underline{T}_1^{\mathrm{NR}})$, and $c(\mathrm{can}^*_{T_i,L}) = e(L/K)\cdot c(T_i)$, $i = 1, 2$. The lemma follows. ∎

§11. Isogeny invariance in characteristic 0

(11.1) It would be useful to prove that the invariant $c(T)$ does not change under isogeny when the local field K has characteristic 0. When the residue field also has characteristic zero, this is not difficult: There is only tame ramification, and the desired result follows from an application of [Ed, Thm. 4.2].

On the other hand when the residue field of $\mathcal{O}_K$ has characteristic $p > 0$, the isogeny invariance can be proved using Galois cohomology. We offer two proofs. The first one is an arithmetic proof, using Tate's Euler-Poincaré characteristic formula, and is valid when the residue field is finite. The second proof is more general and more geometric; but the idea is the same.

The lemma below may be of some interest. It may be convenient but not strictly necessary for the proof of isogeny invariance.

(11.2) Lemma *Let K be a field equipped with a discrete valuation and let T be a torus over K Let T_{s} be the maximal split subtorus of T, and let T_{a} be the quotient torus T/T_{s}. Then the canonical arrows*

$$1 \longrightarrow \underline{T}_{\mathrm{s}}^{\mathrm{NR}} \longrightarrow \underline{T}^{\mathrm{NR}} \longrightarrow \underline{T}_{\mathrm{a}}^{\mathrm{NR}} \longrightarrow 1$$

form a short exact sequence.

Proof. Writing T_{s} as a successive extension of $\mathbb{G}_{\mathrm{m}}$'s, we may and do assume that $T' \cong \mathbb{G}_{\mathrm{m}}$. As in the proof of [BLR, 7.5/Prop. 1], it suffices to show that the extension

$$1 \longrightarrow T_{\mathrm{s}} \longrightarrow T \longrightarrow T_{\mathrm{a}} \longrightarrow 1$$

extends to an exact sequence of smooth group schemes

$$1 \longrightarrow \underline{T}_{\mathrm{s}}^{\mathrm{NR}} \longrightarrow \underline{T}^{*} \longrightarrow \underline{T}_{\mathrm{a}}^{\mathrm{NR}} \longrightarrow 1$$

for some smooth model $\underline{T}^*$ of T. But this is exactly the statement of [SGA7I, VIII Cor. 6.6]. ∎

(11.3) For the rest of this section, K denotes a local field with mixed characteristic $(0, p)$ and perfect residue field. The theorem below was stated in [G, Prop. 4.7] when the residue field of K is finite.

Theorem *Let T_1, T_2 be two tori over K, and let $\alpha : T_1 \to T_2$ be a K-isogeny. Then the two tori have the same invariant:*

$$c(T_1) = c(T_2) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q})$$

(11.4) Proposition *Consider the pull-back map $\alpha^* : \omega(\underline{T}_2^{\mathrm{NR}}) \to \omega(\underline{T}_1^{\mathrm{NR}})$ as above. There exists an element $a \in \mathcal{O}_K$, unique up to $\mathcal{O}_K^{\times}$, such that $\mathcal{O}_K \cdot \alpha^*(\omega(\underline{T}_2^{\mathrm{NR}})) = a \cdot \omega(\underline{T}_1^{\mathrm{NR}})$. Denote the rational number $p^{\mathrm{ord}_p(a)}$ by $\deg_{\mathrm{diff}}(\alpha)$, which we call the differential degree of α. Then*

$$\deg_{\mathrm{diff}}(\alpha) \leq p^{\mathrm{ord}_p(\deg \alpha)}$$

In the above, ord_p *denotes the valuation on K with* $\mathrm{ord}_p(p) = 1$.

(11.4.1) *Proof of Proposition 11.4 when K is a finite extension of $\mathbb{Q}_p$.*

By Lemma 11.2, we may and do assume that T_1 and T_2 are anisotropic over the maximal unramified extension of K (replacing K by a finite unramified extension if necessary). Then $\underline{T}_i^{\mathrm{NR}}(L) = T_i(L)$ for any unramified extension L/K, $i = 1, 2$.

Let $\underline{T}_1^{\mathrm{NR}\circ}$ (resp. $\underline{T}_2^{\mathrm{NR}\circ}$) be the neutral component of the Néron model $\underline{T}_1^{\mathrm{NR}}$ (resp. $\underline{T}_2^{\mathrm{NR}}$). Let ω_i be an $\mathcal{O}_K$-generator of $\omega(\underline{T}_i^{\mathrm{NR}})$, $i = 1, 2$. Let $n = \mathrm{ord}_K(a) = \mathrm{ord}_K(\deg_{\mathrm{diff}}(\alpha))$. Let $M = \ker(\alpha)$, the kernel of the isogeny α.

Consider finite unramified extensions L of K. Let q_L be the cardinality of the residue field κ_L of $\mathcal{O}_L$. Let $|\omega_2|$ (resp. $|\alpha^*\omega_2|$) be the Haar measure on $\underline{T}_2^{\mathrm{NR}}(\mathcal{O}_L)$ (resp. $\underline{T}_1^{\mathrm{NR}}(\mathcal{O}_L)$) attached to ω_2 (resp. $\alpha^*\omega_2$). Considering the volumes of $\underline{T}_2^{\mathrm{NR}\circ}(\mathcal{O}_L)$ $\underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)$, we have

$$|\alpha^*\omega_2|\Big(\underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\Big) = \mathrm{Card}\Big(M(L) \cap \underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\Big) \cdot |\omega_2|\Big(\alpha\big(\underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\big)\Big)$$

By definition, for $i = 1, 2$, $|\omega_i|\big(\underline{T}_i^{\mathrm{NR}}(\mathcal{O}_L)\big)$ is equal to the number of q_L-rational points of the closed fibre of $\underline{T}_i^{\mathrm{NR}\circ}$, divided by $q_L^{\dim T_i}$. Since T_i is anisotropic, its closed fibre is a unipotent group over κ_L and has the same number of κ_L-rational points as $\mathbb{A}^{\dim(T_i)}$. Hence $|\omega_1|\big(\underline{T}_1^{\mathrm{NR}}(\mathcal{O}_L)\big) = |\omega_2|\big(\underline{T}_2^{\mathrm{NR}}(\mathcal{O}_L)\big)$, and the previous formula gives

$$\Big[\underline{T}_2^{\mathrm{NR}\circ}(\mathcal{O}_L) : \alpha\big(\underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\big)\Big] = \mathrm{Card}\Big(M(L) \cap \underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\Big) \cdot q^n$$

Notice that $\mathrm{Card}\big(M(L) \cap \underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\big)$ become constant if L is large enough.

On the other hand, let C_{T_1} (resp. C_{T_2}) be the group of geometric connected components of the closed fibre of $\underline{T}_1^{\mathrm{NR}}$ (resp. $\underline{T}_2^{\mathrm{NR}}$). For sufficiently large finite unramified extensions L of K, we have

$$\Big[\underline{T}_2^{\mathrm{NR}\circ}(\mathcal{O}_L) : \alpha\big(\underline{T}_1^{\mathrm{NR}\circ}(\mathcal{O}_L)\big)\Big] = \frac{\mathrm{Card}(C_{T_1})}{\mathrm{Card}(C_{T_2})} \cdot \Big[\underline{T}_2^{\mathrm{NR}}(\mathcal{O}_L) : \alpha\big(\underline{T}_1^{\mathrm{NR}}(\mathcal{O}_L)\big)\Big]$$

Combined with the previous paragraph, we see that the differential degree $\deg_{\mathrm{diff}}(\alpha)$ of α can be detected from the growth behavior of

$$\Big[T_2(L) : \alpha\big(T_1(L)\big)\Big] = \Big[\underline{T}_2^{\mathrm{NR}}(\mathcal{O}_L) : \alpha\big(\underline{T}_1^{\mathrm{NR}}(\mathcal{O}_L)\big)\Big]$$

as L runs through finite unramified extensions of K.

On the other hand, by Tate's formula for the Euler-Poincaré characteristic for the Galois cohomologies of local fields (see [S3, II Thm. 5]) we have

$$\mathrm{Card}\big(\mathrm{H}^1(L, M)\big) = q_L^{\mathrm{ord}_K(\deg\alpha)} \cdot \mathrm{Card}\big(M(L)\big) \cdot \mathrm{Card}\big(\mathrm{H}^2(L, M)\big)$$

By the local duality for Galois cohomology of local fields ([M, I Cor. 2.3]), $\mathrm{H}^2(L, M)$ is the dual of $M^D(L)$, where M^D is the Cartier dual of the finite group scheme M over K. This finite group $M^D(L)$ become constant if L is sufficiently large.

From the long exact sequence of Galois cohomologies attached to the isogeny α, we get an injection from $T_2(L)/\alpha(T_1(L))$ to $\mathrm{H}^1(L, M)$. This gives us the required bound

$$\deg_{\mathrm{diff}}(\alpha) \leq p^{\mathrm{ord}_p(\deg\alpha)} \quad \blacksquare$$

(11.4.2) *Proof of Proposition 11.4, the general case.*

We may and do assume that K is complete and the residue field κ of K is algebraically closed. As before, we may and do assume that T_1 and T_2 are anisotropic. Our proof is similar to [R, p.208–210] in flavor.

For $i = 1, 2$ and for each $n > 0$, let $\underline{T}_{i,n}$ be the Greeberg functor applied to $\underline{T}_i^{\rm NR} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathcal{O}^n$. Each $\underline{T}_{i,n}$ is a smooth group scheme over the residue field κ, of dimension $nd = n \dim(T_i)$. We have canonical isomorphisms $\underline{T}_{i,n}(\kappa) = \underline{T}_i(\mathcal{O}^n)$. Let

$$\underline{T}_i = \varprojlim_n \underline{T}_{i,n}$$

where the projective system is regarded as a projective system in the category of quasi-algebraic groups, and the limit is a pro-algebraic group in the sense of Serre [S4]. We have canonical isomorphisms

$$\underline{T}_i(\kappa) = \underline{T}_i^{\rm NR}(\mathcal{O}_K) = T_i(K)$$

The isogeny α induces homomorphisms $\underline{\alpha}_n : \underline{T}_{1,n} \to \underline{T}_{2,n}$ and $\underline{\alpha} : \underline{T}_1 \to \underline{T}_2$. Let $\underline{M}$ (resp. $\underline{C}$) be the kernel (resp. cokernel) of $\underline{\alpha}$. We have an exact sequence

$$0 \to \underline{M} \to \underline{T}_1 \to \underline{T}_2 \to \underline{C} \to 0$$

Let M be the kernal of α, a finite group scheme over K. The kernel $\underline{M}$ is a finite quasi-algebraic group over κ. We claim that the dimension of the quasi-algebraic group $\underline{C}$ is equal to $\deg_{\rm diff}(\alpha)$.

To prove this claim, we proceed as in [R, 2.2.1]. Choose $N \gg 0$ such that for $i = 1, 2$, the exponential map induces an isomorphism from $\pi^N \operatorname{Lie} \underline{T}_i^{\rm NR}$ to $\ker\big(\underline{T}_i^{\rm NR}(\mathcal{O}) \to \underline{T}_i^{\rm NR}(\mathcal{O}^N)\big)$. The group $\pi^N \operatorname{Lie} \underline{T}_i^{\rm NR}$ has an obvious structure of pro-algebraic group $\underline{\pi^N \operatorname{Lie} \underline{T}_i^{\rm NR}}$ over κ. Recall from [S4, 2.4, Prop. 7] that the category of pro-algebraic groups is abelian. The following map between short exact sequences of pro-algebraic groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\pi^N \operatorname{Lie} \underline{T}_1^{\rm NR}} & \longrightarrow & \underline{T}_1 & \longrightarrow & \underline{T}_{1,N} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \underline{\operatorname{Lie}\alpha}} & & \Big\downarrow{\scriptstyle \underline{\alpha}} & & \Big\downarrow{\scriptstyle \underline{\alpha}_N} & & \\
0 & \longrightarrow & \underline{\pi^N \operatorname{Lie} \underline{T}_2^{\rm NR}} & \longrightarrow & \underline{T}_2 & \longrightarrow & \underline{T}_{2,N} & \longrightarrow & 0
\end{array}$$

gives rise to an exact sequence

$$0 \to \underline{M} \to \ker(\underline{\alpha_N}) \to \operatorname{coker}(\underline{\operatorname{Lie}\alpha}) \to \underline{C} \to \operatorname{coker}(\underline{\alpha_N}) \to 0$$

We deduce from the exact sequence that $\dim \underline{C} = \dim \operatorname{coker}(\underline{\operatorname{Lie}\alpha})$, which is equal to the length of the cokernel of $\pi^N \operatorname{Lie}(\underline{T}_1^{\rm NR}) \to \pi^N \operatorname{Lie}(\underline{T}_2^{\rm NR})$. This proves the claim.

By [Be, 4.3.3], the cohomology group $\mathrm{H}^1(K, M)$ has a natural structure as a quasi-algebraic group $\underline{\mathrm{H}^1(K, M)}$ over κ of dimension $\operatorname{ord}_p(\deg \alpha)$. The long exact sequence attached to

$$0 \to M \to T_1 \to T_2 \to 0$$

gives us an injection from $\underline{C}$ into $\underline{\mathrm{H}^1(K, M)}$. Hence $\dim \underline{C} \le \dim \underline{\mathrm{H}^1(K, M)}$. The proposition is proved. ∎

(11.5) *Proof of Theorem 11.3.*

By Prop. 10.2, it suffices to show $c(T_1) = c(T_2)$. This follows easily from the bound provided by Prop 11.4: Choose an isogeny $\beta : T_2 \to T_1$ such that $\beta \circ \alpha = [n]_{T_1}$ for some integer $n > 0$. Write $n = p^a \cdot u$, where $a = \mathrm{ord}_p(n), u \in \mathbb{Z}$, and let $d = \dim T_1 = \dim T_2$. We have

$$p^{ad} = \deg_{\mathrm{diff}}(\beta \circ \alpha) = \deg_{\mathrm{diff}}(\beta)\ \deg_{\mathrm{diff}}(\alpha) \leq p^{\mathrm{ord}_p(\deg \alpha)} p^{\mathrm{ord}_p(\deg \beta)} = p^{ad}$$

So the equality holds throughout the above displayed inequality. This implies that $c(T_1) = c(T_2)$ and the theorem is proved. ∎

§12. Isogeny invariance in characteristic p

(12.1) Theorem *Assume that K is of equal-characteristic p and the residue field of $\mathcal{O}_K$ is perfect. Let T be a torus over K. Then*

$$c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q}).$$

In particular, it is invariant under isogeny.

Proof. Without loss of generalities, we may replace K by its completion. So we now assume that K is complete.

From the definition, it is clear that the invariant $c(T)$ is determined by the morphism

$$\underline{T}^{\mathrm{NR}} \otimes \mathcal{O}_L^m \to \underline{(T \otimes L)}^{\mathrm{NR}} \otimes \mathcal{O}_L^m$$

for $m \gg 0$. By the discussion of (9.3), it follows that $c(T)$ is determined by $(\mathrm{Tr}_e K, \Lambda)$ for $e \gg 0$.

Choose e large enough such that $c(T)$ is determined by $(\mathrm{Tr}_e K, \Lambda)$. Choose a local field K_0 of characteristic 0 such that $\mathrm{Tr}_e K_0 \simeq \mathrm{Tr}_e K$. By applying Theorem 11.3 to the torus over K_0 determined by $(\mathrm{Tr}_e K_0, \Lambda)$, we see that $c(T) = c(T_0)$ is one-half of the Artin conductor of Λ as a $\mathrm{Gal}(K_0^{\mathrm{sep}}/K_0)$-module. This Artin conductor is determined by Λ as a module over $\Gamma(\mathrm{Tr}_e K_0) = \Gamma(\mathrm{Tr}_e K)$ and the upper numbering filtration on $\Gamma(\mathrm{Tr}_e K_0) = \Gamma(\mathrm{Tr}_e K)$, and is equal to the Artin conductor of Λ as a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module. The theorem is proved. ∎

(12.2) Remark Let $0 \leq c_1 \leq \cdots \leq c_n$ be the exponents of the elementary divisors of the following torsion module

$$\frac{\mathrm{Lie}\, \underline{(T \otimes L)}^{\mathrm{NR}}}{(\mathrm{can}_{T,L})_*(\mathrm{Lie}\, \underline{T}^{\mathrm{NR}}) \otimes \mathcal{O}_L},$$

divided by $e(L/K)$. Then each $c_i = c_i(T)$ is an invariant of $T_{/K}$ and $c(T) = \sum c_i(T)$. Our main theorem shows that $c_i(T)$ depends only on $(T_e(K), \Lambda)$ for $e \gg 0$. But we don't know anything else about these invariants.

Appendix : Another approach to the characteristic p case

by Ehud de Shalit

§A1. A recipe for the Lie algebra

In the appendix we describe a second approach to Theorem 12.1. It is also based on the Deligne-Kazhdan-Krasner idea that local fields in characteristic p are limits of local fields in characteristic 0, as the absolute index of ramification grows to infinity. But it deals directly with the Lie algebra, rather than the Néron model. In other words, if the main idea behind the first approach was to show that the reduction of the Néron model modulo π^N depended functorially only on some "truncated data", we do the same here with the Lie algebra. Although the result is weaker, it is sufficient to prove Theorem 12.1 (which motivated this work), and the proof is somewhat simpler.

As before, let K be a local field, T a torus over K, and L a finite Galois extension of K splitting T. Let $\Gamma = \mathrm{Gal}(L/K)$. We proceed in three steps. In the first, outlined in the present section, we obtain a recipe for the Lie algebra of the Néron model of T in terms of the character group $M = X^*(T)$ of the torus, as a Γ-module. This recipe is of independent interest, and does not seem to follow from the previous sections. In the second step, we extend some old results of S. Sen [Sen] on the Galois cohomology of local fields in characteristic 0. Then, in Prop. A3.2, we show that the invariant $c(T)$ depends on "truncated data" only. Our Sen-type result implies that in characteristic 0, *the depth of the truncation depends on* M, the Galois group Γ, and its ramification filtration, but *not on the actual local field* K. This "uniformity" result in Galois cohomology is essential in the last step, where we apply Deligne's method to reduce the main theorem in characteristic p, to the characteristic 0 case (as in Theorem 11.3 above).

It is interesting to note that in addition to Deligne's theory, there are other points of contact between the two approaches. For example, the use of Elkik's result in the first approach, finds an analogue in the second, both in our use of M. Artin's approximation theorem in Lemma A1.6 below, and later on, when we make precise the depth of the truncation on which $c(T)$ depends.

The appendix is logically independent of §3–§12, except for admitting Theorem 11.3 (the main theorem in characteristic 0) as an input. We tried to make it notationally self-contained too, although cross-references are inevitable.

(A1.1) Generalities on tori As in §2, we consider a local field K, a separable closure K^{sep}, and a finite Galois extension L of K, contained in K^{sep}. Then $\Gamma = \mathrm{Gal}(L/K)$ is a quotient of $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We let $G \times \Gamma$ act on Γ on the left via $(g \times \gamma)(\sigma) = g\sigma\gamma^{-1}$ $(g \in G,\ \gamma, \sigma \in \Gamma)$. We let $G \times \Gamma$ act on K^{sep} via the projection to G. We then have a canonical ring-isomorphism, respecting the action of $G \times \Gamma$,

$$K^{\mathrm{sep}} \otimes_K L \simeq \mathrm{Maps}(\Gamma, K^{\mathrm{sep}}) \tag{A1.1}$$

sending $a \otimes b$ to the map $\sigma \mapsto a \cdot \sigma(b)$.

Let T be a torus over K, split by L, and

$$M = X^*(T) = \mathrm{Hom}(T_{/K^{\mathrm{sep}}}, \mathbb{G}_{m/K^{\mathrm{sep}}}) \tag{A1.2}$$

the character group. As a Γ-module, M is the dual lattice of $\Lambda = X_*(T)$. As functors on K-algebras we have the canonical identification

$$T(-) = \mathrm{Hom}_\Gamma(M, (- \otimes_K L)^\times).$$

Indeed, the right hand side defines a torus T_1 whose character group is $(M \otimes \mathbb{Z}\Gamma)_\Gamma$ (coinvariants) and the map $m \otimes \sigma \mapsto \sigma m$ is an isomorphism $(M \otimes \mathbb{Z}\Gamma)_\Gamma \simeq M$ carrying the G action on the first to the original Γ action on the second. Therefore T_1 is the original T.

Let $T' = T \times_{\operatorname{Spec} K} \operatorname{Spec} L$. Then $T'(-) = \operatorname{Hom}(M, (-)^\times)$ is the canonical trivialization of T over L.

(A1.2) Néron models We need a watered-down version of the results explained in detail in Section 3. Let $S = \operatorname{Spec}(\mathcal{O}_K)$ and $S' = \operatorname{Spec}(\mathcal{O}_L)$. Let $\mathcal{N}$ (resp. $\mathcal{N}'$) be the Néron (lft-)model of T (resp. T') over S (resp. S') and let $\mathcal{T} = \mathcal{N}^0$ (resp. $\mathcal{T}' = \mathcal{N}'^0$) be its neutral component.

(A1.3) Proposition ([BLR, 10.1/4]) *$\mathcal{N}$ is the* group smoothening *of the schematic closure of T in the Weil restriction of scalars $\operatorname{Res}_{S'/S} \mathcal{N}'$.*

See [BLR, p.174] for the notion of group smoothening. It is defined by a universal property, but in practice is carried out by successive dilatations as described in §4, each time reducing Néron's defect of smoothness, until we reach a smooth group scheme.

Since T' is split, we have, as functors on $\mathcal{O}_L$- and $\mathcal{O}_K$-algebras, the identities

$$\mathcal{T}'(-) = \operatorname{Hom}(M, (-)^\times), \quad \operatorname{Res}_{S'/S} \mathcal{T}'(-) = \operatorname{Hom}(M, (- \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times). \tag{A1.3}$$

We shall make use of the intermediate model $\mathcal{X}$, defined as a functor on $\mathcal{O}_K$-algebras by

$$\mathcal{X}(-) = \operatorname{Hom}_\Gamma(M, (- \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times), \tag{A1.4}$$

and we let $\mathcal{Y}$ stand for the schematic closure of T in $\mathcal{X}$. Of course, $\mathcal{Y}_K = \mathcal{X}_K = T$, and $\mathcal{Y}$ is a closed subscheme of $\mathcal{X}$.

In concrete terms, $\operatorname{Res}_{S'/S} \mathcal{T}' = \operatorname{Spec} A$, $T = \operatorname{Spec} A_K/I$ and the ideal I is generated, in A_K, by a collection $\{f_i\}$ of polynomials expressing the condition that a homomorphism from M to $(- \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times$ be Γ-invariant. Introducing a $\mathbb{Z}$-basis in M, it is clear that these polynomials are in fact in A. Then $\mathcal{X} = \operatorname{Spec} A/\{f_i\}$, and $\mathcal{Y} = \operatorname{Spec} A/\mathcal{I}$, where $\mathcal{I} = \ker(A \to A_K/I)$. The $\{f_i\}$ may not generate $\mathcal{I}$ over A, because there could be an A_K-linear combination of them lying in A, which is not an A-linear combination. However, the following lemma means that for our purposes the difference between $\mathcal{Y}$ and $\mathcal{X}$ is unimportant.

(A1.4) Lemma *For any $\mathcal{O}_K$-algebra R which* embeds *in R_K, $\mathcal{Y}(R) = \mathcal{X}(R)$.*

Proof. Letting A, I and $\mathcal{I}$ be as above, take $x \in \mathcal{X}(R)$, and let x_K be its generic fiber. Then we have a diagram

$$\begin{tikzcd} I \arrow[r, hook] & A_K \arrow[r, "x_K"] & R_K \\ \mathcal{I} \arrow[u] \arrow[r, hook] & A \arrow[u] \arrow[r, "x"] & R \arrow[u, hook] \end{tikzcd} \tag{A1.5}$$

and $x_K(I) = 0$. It follows that $x(\mathcal{I}) = 0$, and x factors through $\mathcal{Y}$.

Since $\mathcal{X}$ is a closed subscheme of $\mathrm{Res}_{S'/S}\,\mathcal{T}' = \mathrm{Res}_{S'/S}(\mathcal{N}'^0)$, $\mathcal{Y}$ is the schematic closure of T in $\mathrm{Res}_{S'/S}(\mathcal{N}'^0)$, so (by Prop. A1.3) $\mathcal{T}$ is the neutral component of the group smoothening of $\mathcal{Y}$. (Note that even though the special fiber of $\mathcal{Y}$ is connected, the special fiber of its group smoothening need not be connected.)

(A1.5) Lie algebras If $R = \mathcal{O}_K[[t]]$ or $R = \mathcal{O}_K[\varepsilon]$ ($\varepsilon = t \bmod t^2$, $\varepsilon^2 = 0$), we let, for any S-group scheme $\mathcal{G}$

$$\mathcal{G}(R)_1 = \ker\left(\mathcal{G}(R) \to \mathcal{G}(\mathcal{O}_K)\right). \tag{A1.6}$$

We also write R_1 for $\mathbb{G}_m(R)_1 = \ker(R^\times \to \mathcal{O}_K^\times)$. Clearly $\mathcal{G}(R)_1 \subset \mathcal{G}^0(R)$. If $\mathcal{G}$ is smooth over S, then by definition

$$\mathrm{Lie}\,\mathcal{G} = \mathcal{G}(\mathcal{O}_K[\varepsilon])_1. \tag{A1.7}$$

Consider the diagram

$$\begin{tikzcd}
\mathcal{T}(O_K[[t]])_1 \arrow[r, "\tilde{u}"] \arrow[d] & \mathcal{Y}(\mathcal{O}_K[[t]])_1 \arrow[r, equal] \arrow[d] & \mathrm{Hom}_\Gamma(M, \mathcal{O}_L[[t]]_1) \arrow[d] \\
\mathrm{Lie}\,\mathcal{T} = \mathcal{T}(\mathcal{O}_K[\varepsilon])_1 \arrow[r, hook, "u"] & \mathcal{Y}(\mathcal{O}_K[\varepsilon])_1 \arrow[r, equal] & \mathrm{Hom}_\Gamma(M, \mathcal{O}_L).
\end{tikzcd} \tag{A1.8}$$

We make the following observations regarding the diagram.

- The *left vertical arrow* is surjective because $\mathcal{T}$ is smooth ([BLR], 2.2/6).
- The maps u and $\tilde{u}$ are the group smoothening maps.
- The *equality signs* on the right are based on the last lemma. In the bottom right corner we identified $\mathcal{O}_L[\varepsilon]_1 \simeq \mathcal{O}_L\varepsilon \simeq \mathcal{O}_L$.
- The map u is *injective* because both $\mathrm{Lie}\,\mathcal{T}$ and $\mathrm{Hom}_\Gamma(M, \mathcal{O}_L)$ are $\mathcal{O}_K$-submodules of $\mathrm{Lie}\,T = \mathrm{Hom}_\Gamma(M, L)$.

(A1.6) Lemma *The map $\tilde{u}$ is an isomorphism.*

Proof. The injectivity of $\tilde{u}$ is proved as that of u, and in fact is not needed below. To show that $\tilde{u}$ is *surjective*, let $A_0 = \mathcal{O}_K[t]_{(\pi,t)}$, a local ring. Then

$$A_0 \subset A = A_0^h \subset \widehat{A} = \mathcal{O}_K[[t]], \tag{A1.9}$$

where A is the henselization of A_0, and $\widehat{A}$ its completion. By M. Artin's approximation theorem ([BLR], 3.6/16) any $\widehat{y} \in \mathcal{Y}(\widehat{A})$ can be approximated in the (π, t)-adic topology arbitrarily well by $y \in \mathcal{Y}(A)$. But A is the direct limit of smooth $\mathcal{O}_K$-algebras, and for any smooth $\mathcal{O}_K$-algebra R, $\mathcal{T}(R) = \mathcal{Y}(R)$ ([BLR], 7.1/5). Thus y lifts to a point of $\mathcal{T}(A)$, and we deduce that $\mathrm{Im}(\tilde{u})$ is dense. As $\mathcal{O}_K[[t]]$ is compact and $\tilde{u}$ is continuous in the (π, t)-adic topology, $\mathrm{Im}(\tilde{u})$ is also closed, so $\tilde{u}$ is surjective.

The discussion above leads immediately to our "recipe" for the Lie algebra of T.

(A1.7) Proposition (i) *Canonically and functorially*

$$\mathrm{Lie}(\mathcal{T}) = \big\{ v \in \mathrm{Hom}_\Gamma(M, \mathcal{O}_L) \;\big|\; v \textit{ lifts to } \mathrm{Hom}_\Gamma(M, \mathcal{O}_L[[t]]_1) \big\} \tag{A1.10}$$

as a lattice in $\mathrm{Lie}(T) = \mathrm{Hom}_\Gamma(M, L)$.

(ii) *By the universal property of* $\mathcal{N}'$ *there is a map* $\mathcal{T} \times_S S' \to \mathcal{T}'$. *The associated map on tangent spaces is*

$$\begin{tikzcd} \mathrm{Lie}(\mathcal{T}) \otimes_{\mathcal{O}_K} \mathcal{O}_L \arrow[r, hook] \arrow[d, phantom, "\bigcap"] & \mathrm{Lie}(\mathcal{T}') \arrow[d, equal] \\ \mathrm{Hom}_\Gamma(M, \mathcal{O}_L) \otimes_{\mathcal{O}_K} \mathcal{O}_L \arrow[r, hook] & \mathrm{Hom}(M, \mathcal{O}_L). \end{tikzcd} \tag{A1.11}$$

(A1.8) Example: Restriction of scalars Consider $T = \mathrm{Res}_{L/K}\, \mathbb{G}_m$. Then $M = \mathbb{Z}\Gamma$. It follows that $\mathrm{Hom}_\Gamma(M, \mathcal{O}_L) = \mathcal{O}_L$, and every v lifts, so the inclusion $\mathrm{Lie}(\mathcal{T}) \otimes_{\mathcal{O}_K} \mathcal{O}_L \hookrightarrow \mathrm{Lie}(\mathcal{T}')$ is the inclusion $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \hookrightarrow \mathrm{Hom}(\mathbb{Z}\Gamma, \mathcal{O}_L)$ sending $a \otimes b$ to the homomorphism $\sigma \mapsto \sigma(a) \cdot b$ ($\sigma \in \Gamma$). Let δ_σ be the homomorphism which is 1 on σ and 0 on $\tau \in \Gamma$, $\tau \neq \sigma$, and let $\{\alpha\}$ be a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$. Then in terms of the $\mathcal{O}_L$-bases $\{\alpha \otimes 1\}$ and $\{\delta_\sigma\}$ the matrix of the inclusion $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \hookrightarrow \mathrm{Hom}(\mathbb{Z}\Gamma, \mathcal{O}_L)$ is given by $(\sigma(\alpha))$. Its determinant, up to a unit, is $\sqrt{\Delta_{L/K}}$, the square root of the discriminant. Let $a(M_\mathbb{Q})$ be the (exponent of the) Artin conductor of the rational representation $M_\mathbb{Q}$. Then by the conductor-discriminant formula $a(M_\mathbb{Q}) = a(\mathbb{Q}\Gamma) = \mathrm{ord}_K(\Delta_{L/K})$, where ord_K is the normalized valuation on K. It follows that in this case the invariant $c(T)$ defined in §10.1 is equal to $a(M_\mathbb{Q})/2$. This verifies the main theorem for tori which are restriction of scalars, or products of such.

(A1.9) The invariant $c(T)$ It is instructive to compute more examples using Prop. A1.7, and note which "part" of the invariant $c(T)$ is attributed to the failure to lift every $v \in \mathrm{Hom}_\Gamma(M, \mathcal{O}_L)$ to $\mathrm{Hom}_\Gamma(M, \mathcal{O}_L[[t]]_1)$. For lack of space we do not do it here. In general, we may write

$$c(T) = c_0(T) + c_{\mathrm{lift}}(T) \tag{A1.12}$$

where

$$c_0(T) = e(L/K)^{-1}\, \mathrm{length}_{\mathcal{O}_L} \left(\mathrm{Hom}(M, \mathcal{O}_L) / \mathrm{Hom}_\Gamma(M, \mathcal{O}_L) \otimes_{\mathcal{O}_K} \mathcal{O}_L\right), \tag{A1.13}$$

and

$$\begin{aligned} c_{\mathrm{lift}}(T) &= e(L/K)^{-1}\, \mathrm{length}_{\mathcal{O}_L} \left(\mathrm{Hom}_\Gamma(M, \mathcal{O}_L) \otimes_{\mathcal{O}_K} \mathcal{O}_L / \mathrm{Lie}(\mathcal{T}) \otimes_{\mathcal{O}_K} \mathcal{O}_L\right) \\ &= \mathrm{length}_{\mathcal{O}_K} \left(\mathrm{Hom}_\Gamma(M, \mathcal{O}_L) / \mathrm{Lie}(\mathcal{T})\right). \end{aligned} \tag{A1.14}$$

To analyze $c_{\mathrm{lift}}(T)$ introduce the following notation. For any commutative ring k and an additive subgroup $I \subset k$ write $R(k) = k[[t]]$ and

$$\begin{cases} R_i(k) = 1 + t^i R(k) \subset R(k)^\times, \\ R_i(I) = 1 + t^i I[[t]], \end{cases} \qquad i \geq 1. \tag{A1.15}$$

The short exact sequence of Γ-modules

$$0 \to R_2(\mathcal{O}_L) \to R_1(\mathcal{O}_L) \to \mathcal{O}_L \to 0, \tag{A1.16}$$

yields, upon applying $\mathrm{Hom}(M, -)$ and taking the long exact sequence in cohomology, an exact sequence

$$0 \to \mathrm{Lie}(\mathcal{T}) \to \mathrm{Hom}_\Gamma(M, \mathcal{O}_L) \to H^1(\Gamma, \mathrm{Hom}(M, R_2(\mathcal{O}_L))). \tag{A1.17}$$

We have taken here into account the characterization of $\mathrm{Lie}(\mathcal{T})$ given by Prop. A1.7.

§A2. Some results in Galois cohomology

Before we proceed we need to develop some old ideas of S.Sen about the Galois cohomology of local fields in characteristic 0 [Sen]. In this section, let E/F be a finite Galois extension of local fields in characteristic 0. Thus F (and E) is a finite extension of $\mathbb{Q}_p$. Let π_F (resp. π_E) be a uniformizer of F (resp. E). The group $\Gamma = \mathrm{Gal}(E/F)$ is equipped with its (lower and upper) ramification filtrations, $\{\Gamma_u\}$ and $\{\Gamma^v\}$. The upper and lower filtrations determine each other, by means of the Herbrand functions (see [S1]). Let M be a Γ-module, free of finite rank over $\mathbb{Z}$. As in §2.4, we shall say that a quantity "X" *depends on* $(M, \Gamma, \mathit{fil})$ if there is a canonical way to compute "X" from knowledge of Γ, the Γ-module M, and the ramification filtrations on Γ. The emphasis here will be on *independence of the actual field extension* E/F. For example, the exponent $a(M_\mathbb{Q})$ of the Artin conductor depends only on $(M, \Gamma, \mathit{fil})$ by its very definition. Note that this is not true for the Artin conductor itself. The more ramified F is, the smaller the Artin conductor becomes.

(A2.1) Lemma *There is an integer ν, depending only on $(M, \Gamma, \mathit{fil})$, such that*

$$H^1(\Gamma, \mathrm{Hom}(M, \mathcal{O}_E))$$

is killed by π_F^ν.

Proof. (i) When $M = \mathbb{Z}$ and Γ is cyclic of prime order, this is Theorem 2 of [Sen].

(ii) When $M = \mathbb{Z}$ but Γ is arbitrary, we reduce immediately to the case where Γ is a p-group. But then Γ is nilpotent, so this case follows from (i) by induction on $|\Gamma|$, using the inflation-restriction exact sequence. (Note that Theorem 3 of [Sen] is not strong enough for our purposes.)

(iii) Let M and Γ be arbitrary. Write $W = \mathrm{Hom}(M, \mathcal{O}_E)$. Let T be the torus over F, split by E, whose character group is M. By Theorem 11.3, *which we assume valid in the characteristic 0 case,*

$$e(E/F)^{-1}\,\mathrm{length}_{\mathcal{O}_E}\left(W/W^\Gamma \otimes_{\mathcal{O}_F} \mathcal{O}_E\right) = c_0(T) \leq c(T) = a(M_\mathbb{Q})/2 \tag{A2.1}$$

so $W/W^\Gamma \otimes_{\mathcal{O}_F} \mathcal{O}_E$ is killed by a power of π_F depending on $(M, \Gamma, \mathit{fil})$ only, and the same is true for $H^1(\Gamma, W/W^\Gamma \otimes_{\mathcal{O}_F} \mathcal{O}_E)$. But the sequence

$$H^1(\Gamma, W^\Gamma \otimes_{\mathcal{O}_F} \mathcal{O}_E) \to H^1(\Gamma, W) \to H^1(\Gamma, W/W^\Gamma \otimes_{\mathcal{O}_F} \mathcal{O}_E) \tag{A2.2}$$

is exact. The first term is equal to $W^\Gamma \otimes_{\mathcal{O}_F} H^1(\Gamma, \mathcal{O}_E)$, which is killed by a power of π_F depending only on $(M, \Gamma, \mathit{fil})$ by case (ii). It was already observed that the same holds for the last term. Hence $H^1(\Gamma, W)$ as well is killed by a power of π_F depending only on $(M, \Gamma, \mathit{fil})$ and the lemma is proved.

(A2.2) Lemma *There is an integer N, depending only on (M, Γ, fil), such that*

$$H^1(\Gamma, \mathrm{Hom}(M, R_i(\mathcal{O}_E))) \hookrightarrow H^1(\Gamma, \mathrm{Hom}(M, R_i(\mathcal{O}_E/\pi_F^N\mathcal{O}_E))) \qquad \text{(A2.3)}$$

is injective for every $i \geq 1$.

Proof. Clearly, if the lemma holds with one N, it holds with any larger N as well. Let ν be as in the previous lemma, and $N \geq 2\nu$. Suppose that $\{\varphi_\sigma\}$ is a 1-cocycle in $\mathrm{Hom}(M, R_i(\mathcal{O}_E))$, which becomes a 1-coboundary in $\mathrm{Hom}(M, R_i(\mathcal{O}_E/\pi_F^N\mathcal{O}_E))$. Modifying it, we may assume to begin with that $\varphi_\sigma \in \mathrm{Hom}(M, R_i(\pi_F^N\mathcal{O}_E))$. If

$$\varphi_\sigma(m) = 1 + a_\sigma(m)t^i + \cdots \quad (m \in M) \qquad \text{(A2.4)}$$

then $\{a_\sigma\}$ is a 1-cocycle in $\mathrm{Hom}(M, \pi_F^N\mathcal{O}_E)$. It follows from the previous lemma that there exists a $b \in \mathrm{Hom}(M, \pi_F^{N-\nu}\mathcal{O}_E)$ such that $a_\sigma = \sigma(b) - b$ (i.e. $a_\sigma(m) = \sigma(b(\sigma^{-1}m)) - b(m)$).

Pick a basis $m_1, \ldots, m_d$ of M, and define $\beta \in \mathrm{Hom}(M, R_i(\pi_F^{N-\nu}\mathcal{O}_E))$ by

$$\beta(\sum_{j=1}^{d} k_j m_j) = \prod_{j=1}^{d}(1 + b(m_j)t^i)^{k_j}. \qquad \text{(A2.5)}$$

Then

- $\beta(m) = 1 + b(m)t^i + \cdots$
- $\beta(m) \equiv 1 + b(m)t^i \pmod{\pi_F^N\mathcal{O}_E}$.

The second point follows from $N \geq 2\nu$. Now $(\sigma(\beta)/\beta)(m) = 1 + a_\sigma(m)t^i + \cdots \equiv 1 \bmod \pi_F^N\mathcal{O}_E$. Modifying the cocycle $\{\varphi_\sigma\}$ by the coboundary $\{\sigma\beta/\beta\}$ we see that the same cohomology class is represented by a cocycle in $\mathrm{Hom}(M, R_{i+1}(\pi_F^N\mathcal{O}_E))$. Continuing in this way we see that the cohomology class of $\{\varphi_\sigma\}$ is trivial, and the map in the lemma is therefore injective.

If $\varphi \in \mathrm{Hom}_\Gamma(M, \mathcal{O}_E)$ we write φ^N for its image in $\mathrm{Hom}_\Gamma(M, \mathcal{O}_E/\pi_F^N\mathcal{O}_E)$.

(A2.3) Corollary *Let L/K be a finite Galois exetension of local fields, $\Gamma = \mathrm{Gal}(L/K)$, and M a Γ-module, free of finite rank over $\mathbb{Z}$. Then there exists an N such that $\varphi \in \mathrm{Hom}_\Gamma(M, \mathcal{O}_L)$ lifts to $\mathrm{Hom}_\Gamma(M, R_1(\mathcal{O}_L))$ if and only if $\varphi^N \in \mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^N\mathcal{O}_L)$ lifts to $\mathrm{Hom}_\Gamma(M, R_1(\mathcal{O}_L/\pi_K^N\mathcal{O}_L))$. Furthermore, in the characteristic 0 case this N depends only on (M, Γ, fil).*

Proof. Let Φ be the subgroup of $\varphi \in \mathrm{Hom}_\Gamma(M, \mathcal{O}_L)$ which lift, Φ^N the subgroup of those φ for which φ^N lifts to $\mathrm{Hom}_\Gamma(M, R_1(\mathcal{O}_L/\pi_K^N\mathcal{O}_L))$, and $\bar{\Phi}^N$ the subgroup of $\mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^N\mathcal{O}_L)$ which lift to $\mathrm{Hom}_\Gamma(M, R_1(\mathcal{O}_L/\pi_K^N\mathcal{O}_L))$. From the discussion in §A1.9, and from the analogous discussion modulo π_K^N we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi & \longrightarrow & \mathrm{Hom}_\Gamma(M, \mathcal{O}_L) & \longrightarrow & H^1(\Gamma, \mathrm{Hom}(M, R_2(\mathcal{O}_L))) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{\Phi}^N & \longrightarrow & \mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^N\mathcal{O}_L) & \longrightarrow & H^1(\Gamma, \mathrm{Hom}(M, R_2(\mathcal{O}_L/\pi_K^N\mathcal{O}_L))), \end{array} \qquad \text{(A2.6)}$$

in which the rows are exact. In the characteristic 0 case, the last lemma shows that for N large enough, *depending only on* $(M, \Gamma, \mathit{fil})$, the last vertical arrow is injective. Thus $\varphi \in \Phi$ if and only if $\varphi^N \in \bar{\Phi}^N$.

In general, Proposition A1.7 identifies $\Phi = \operatorname{Lie} \mathcal{T}$ and it follows that Φ is of finite index in $\operatorname{Hom}_\Gamma(M, \mathcal{O}_L)$, because both are $\mathcal{O}_K$-lattices of the same rank. Since

$$\Phi \subset \Phi^{N+1} \subset \Phi^N \subset \operatorname{Hom}_\Gamma(M, \mathcal{O}_L), \tag{A2.7}$$

the sequence of subgroups Φ^N stabilizes for large N. But it is clear, by compactness, that if φ is such that φ^N lifts for every N, then φ itself lifts. Therefore $\Phi = \Phi^N$ for large enough N, and the claim is proved.

§A3. Reduction of the char. p case to the char. 0 case

(A3.1) The invariant $c(T)$ depends only on truncated data Let L/K, Γ and M be as in the last corollary, and let T be the torus over K, split by L, whose character group is M. Let $\mathcal{T}$ be as before, the neutral component of the Néron model of T. Choose m large enough such that the following conditions hold.

M1 π_K^m annihilates $\operatorname{Hom}(M, \mathcal{O}_L)/\operatorname{Lie}(\mathcal{T}) \otimes_{\mathcal{O}_K} \mathcal{O}_L$,

M2 If $\operatorname{char.} K = 0$, π_K^m kills $H^1(\Gamma, \operatorname{Hom}(M, \mathcal{O}_L))$. If $\operatorname{char.} K = p$, we require only that the map

$$H^1(\Gamma, \operatorname{Hom}(M, \mathcal{O}_L))[\pi_K^m] \hookrightarrow H^1(\Gamma, \operatorname{Hom}(M, \mathcal{O}_L))/\pi_K^m H^1(\Gamma, \operatorname{Hom}(M, \mathcal{O}_L)) \tag{A3.1}$$

is injective. Note that for any finitely generated $\mathcal{O}_K$-module Λ, $\Lambda[\pi_K^m] \hookrightarrow \Lambda/\pi_K^m \Lambda$ is injective for large enough m.

M3 An element φ of $\operatorname{Hom}_\Gamma(M, \mathcal{O}_L)$ lifts to $\operatorname{Hom}_\Gamma(M, R_1(\mathcal{O}_L))$ if and only if φ^m lifts to $\operatorname{Hom}_\Gamma(M, R_1(\mathcal{O}_L/\pi_K^m \mathcal{O}_L))$.

By Theorem 11.3, Lemma A2.2 and the last corollary, if $\operatorname{char.} K = 0$, then how large m has to be depends only on $(M, \Gamma, \mathit{fil})$.

(A3.2) Proposition *The invariant $c(T)$ depends only on $(M, \Gamma, \mathit{fil})$ and $\mathcal{O}_L/\pi_K^{2m}\mathcal{O}_L$, viewed as a Γ-module. More precisely, let there be given two Galois extensions L/K and E/F of local fields, and an isomorphism $\operatorname{Gal}(L/K) \simeq \operatorname{Gal}(E/F)$ preserving the ramification filtrations. Call this common group Γ and endow it with the resulting filtrations. Let M be a Γ-module, free of finite rank over $\mathbb{Z}$. Let T_K and T_F be the corresponding tori. Let m be an integer for which the above conditions hold, both for L/K and for E/F. Suppose that there is an isomorphism of Γ-modules*

$$\mathcal{O}_L/\pi_K^{2m}\mathcal{O}_L \simeq \mathcal{O}_E/\pi_F^{2m}\mathcal{O}_E. \tag{A3.2}$$

Then $c(T_K) = c(T_F)$.

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_L \arrow[r, "\pi_K^m"] \arrow[d] & \mathcal{O}_L \arrow[r, "\mathrm{can}"] \arrow[d] & \mathcal{O}_L/\pi_K^m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_L/\pi_K^m \arrow[r, "\pi_K^m"] & \mathcal{O}_L/\pi_K^{2m} \arrow[r, "\mathrm{can}"] & \mathcal{O}_L/\pi_K^m \arrow[r] & 0.
\end{tikzcd} \tag{A3.3}$$

Its top row yields a short exact sequence

$$0 \to \mathrm{Hom}_\Gamma(M, \mathcal{O}_L)/\pi_K^m \to \mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^m) \to H^1(\Gamma, \mathrm{Hom}(M, \mathcal{O}_L))[\pi_K^m] \to 0, \tag{A3.4}$$

and also an injection

$$H^1(\Gamma, \mathrm{Hom}(M, \mathcal{O}_L))/\pi_K^m \hookrightarrow H^1(\Gamma, \mathrm{Hom}(M, \mathcal{O}_L/\pi_K^m)). \tag{A3.5}$$

By **M2** we find a short exact sequence

$$0 \to \mathrm{Hom}_\Gamma(M, \mathcal{O}_L)/\pi_K^m \to \mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^m) \xrightarrow{\delta_m} H^1(\Gamma, \mathrm{Hom}(M, \mathcal{O}_L/\pi_K^m)). \tag{A3.6}$$

The second row of the diagram shows that δ_m is computed from knowledge of $\mathcal{O}_L/\pi_K^{2m}$ alone. Thus

$$\mathrm{Hom}_\Gamma(M, \mathcal{O}_L)/\pi_K^m = \ker(\delta_m), \tag{A3.7}$$

as a subgroup of $\mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^m)$, depends on $\mathcal{O}_L/\pi_K^{2m}$ only.

Let Φ_m be the image of $\mathrm{Lie}(\mathcal{T})$ in $\ker(\delta_m)$. By **M1**, $\mathrm{Lie}(\mathcal{T}) \supset \pi_K^m \mathrm{Hom}_\Gamma(M, \mathcal{O}_L)$, so

$$[\mathrm{Hom}_\Gamma(M, \mathcal{O}_L) : \mathrm{Lie}(\mathcal{T})] = [\ker(\delta_m) : \Phi_m]. \tag{A3.8}$$

By **M3**, $\Phi_m = \ker(\delta_m) \cap \ker(\varepsilon_m)$, where

$$\varepsilon_m : \mathrm{Hom}_\Gamma(M, \mathcal{O}_L/\pi_K^m) \to H^1(\Gamma, \mathrm{Hom}(M, R_2(\mathcal{O}_L/\pi_K^m))) \tag{A3.9}$$

is the connecting homomorphism as in the second row of (A2.6). We conclude that $c_{\mathrm{lift}}(T)$, which is computed from the index

$$[\ker(\delta_m) : \ker(\delta_m) \cap \ker(\varepsilon_m)], \tag{A3.10}$$

depends only on $\mathcal{O}_L/\pi_K^{2m}$.

Next, to compute $c_0(T)$ we need to know the index

$$[\mathrm{Hom}(M, \mathcal{O}_L) : \mathrm{Hom}_\Gamma(M, \mathcal{O}_L) \otimes_{\mathcal{O}_K} \mathcal{O}_L], \tag{A3.11}$$

which, by **M1**, is equal to

$$[\mathrm{Hom}(M, \mathcal{O}_L/\pi_K^m) : \ker(\delta_m) \otimes_{\mathcal{O}_K/\pi_K^m} \mathcal{O}_L/\pi_K^m]. \tag{A3.12}$$

This last formula shows that it, too, depends only on $\mathcal{O}_L/\pi_K^{2m}$.

(A3.3) Conclusion of the proof We are now ready to apply the Deligne-Kazhdan-Krasner idea. We use, as in §9, Deligne's formalism. Recall that we assume the validity of Theorem 11.3, and want to derive Theorem 12.1.

Fix a local field K, $\operatorname{char} K = p$, a torus $T = T_K$ over K, and let L/K be a Galois splitting field. Then $\Gamma = \operatorname{Gal}(L/K)$ is equipped with its ramification filtrations. Let $M = X^*(T)$ be the character group of T, a $\mathbb{Z}$-lattice with a Γ-action.

Choose m large enough such that the three conditions on m at the beginning of this section hold for the particular extension L/K, and *also for any extension of local fields E/F in characteristic 0, for which*

$$(\operatorname{Gal}(E/F), \mathit{fil}) \simeq (\Gamma, \mathit{fil}). \tag{A3.13}$$

This is possible by the remark preceding the proposition.

Let e be large enough so that $\Gamma^e = 1$. Let F be any local field of characteristic 0 such that, in Deligne's category of truncated valuation rings,

$$\operatorname{Tr}_e(F) \simeq \operatorname{Tr}_e(K). \tag{A3.14}$$

(See [D], or the introduction, for the notion of "truncation"). Such an F always exists. One simply has to take the absolute index of ramification of F large enough. According to the main theorem of Deligne ([D], Th. 2.8), fixing such an isomorphism induces an equivalence of categories

$$(\operatorname{ext} F)^e \approx (\operatorname{ext} \operatorname{Tr}_e(F))^e \approx (\operatorname{ext} \operatorname{Tr}_e(K))^e \approx (\operatorname{ext} K)^e, \tag{A3.15}$$

between the category of at-most e-ramified extensions of F and the same category for K. In particular there exists then a unique Galois extension E/F corresponding to L/K, and an isomorphism *preserving the ramification filtrations,*

$$\operatorname{Gal}(E/F) \simeq \Gamma = \operatorname{Gal}(L/K). \tag{A3.16}$$

Let $e(1)$ be computed from e as in [D] "Construction 3.4.1". Note that it depends only on (Γ, fil) and e. If e is large enough so that

$$\pi_F^{2m} | \pi_E^{e(1)} \tag{A3.17}$$

then there is a Γ-isomorphism

$$\mathcal{O}_E/\pi_F^{2m}\mathcal{O}_E \simeq \mathcal{O}_L/\pi_K^{2m}\mathcal{O}_L \tag{A3.18}$$

(loc.cit.).

Let T_F be the torus over F, split by E, whose character group is M, viewed as a $\operatorname{Gal}(E/F)$-module via the above identification of the Galois group with Γ. Prop. A3.2 now, finally, implies

$$c(T_K) = c(T_F) = a(M_{\mathbb{Q}})/2, \tag{A3.19}$$

and the proof of Theorem 12.1 is concluded.

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