THE NATURALITY IN KIRWAN'S DECOMPOSITION

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Contents

0. Introduction 1
1. The geometric case 1
2. The arithmetic version 5
References 6

0. Introduction

Let X be a compact simplectic manifold, K a compact group acting simplectically on X, and assume there is a moment map. In [3], Kirwan defined a stratification on X, and showed that the associated spectral sequence degenerates. In this note, we ask whether this degeneration comes from a stable splitting of X into Thom spaces on the strata. One can also ask a related arithmetic question. Is it true that the splitting Kirwan discovered respects the action of Galois groups?

1. The geometric case

Let X be a compact, simplectic manifold. That is, X is a compact manifold, with a symplectic form ω. Suppose K is a compact group acting on X, and suppose that the action preserves the simplectic form. That is, for every $k \in K$, $k^*\omega = \omega$. Suppose further that there is a moment map for the action. We remind the reader what a moment map is. Let $\mathfrak{K}$ is the Lie algebra of K. Let $\mathfrak{K}^*$ be the dual of the Lie algebra. A *moment map* is a map $\mu : X \longrightarrow \mathfrak{K}^*$, which is K–equivariant (the action of K on $\mathfrak{K}^*$ being the coadjoint action). Furthermore, the derivative of μ must satisfy the following identity, where ψ is any vector field on X and a is an element of $\mathfrak{K}$, the Lie algebra of K

$$\{d\mu(\psi)\}(a) = \omega(\psi, a).$$

Throughout the first Section, K and X will be assumed to have all the structure above, without further mention. By the work of Kirwan (Theorem 4.16 of [3]), we know that the manifold X has a natural stratification. There is a finite, partially ordered set B, and for each $\beta \in B$ a locally closed submanifold X_β of X. The submanifolds X_β are all K–invariant; $KX_\beta = X_\beta$. Furthermore,

$$X = \bigcup_{\beta \in B} X_\beta$$

expresses X as the disjoint union of the strata X_β. Finally, suppose we choose a subset $B' \subset B$ which is *initial*, that is whenever $\beta \in B'$ and $\beta' \leq \beta$, we must have $\beta' \in B'$. Then the union

$$X = \bigcup_{\beta \in B'} X_\beta$$

must be open in X. In particular, the lowest strata, the strata corresponding to minimal β's, are all open.

Notation 1.1. The sets we refer to as X_β would, in Kirwan's notation, be called S_β, or more precisely they are the $S_{\beta,m}$'s.

For any space Y with a K–action, let $Y \times_K EK$ be the homotopy quotient of Y by K. That is, if EK is a contractible space with a free K–action, then K acts diagonally on $Y \times EK$, and the quotient by this action is the homotopy quotient, denoted $Y \times_K EK$.

With all the notations as above, the homotopy quotient of X by K is a stratified space, with strata the homotopy quotients of X_β by K. That is, we have

$$\{X \times_K EK\} = \bigcup_{\beta \in B} \{X_\beta \times_K EK\}$$

Whenever we have a stratified space, there is a corresponding spectral sequence for its (co)homology. In fact, let us be general. Suppose E^* is a generalised cohomology theory. There is a spectral sequence for $E^*(X \times_K EK)$, whose E_2 term involves the $E^*(T_\beta)$'s, where T_β is the Thom space on the normal bundle of the subspace

$$X_\beta \times_K EK \subset X \times_K EK.$$

Note that although both $X_\beta \times_K EK$ and $X \times_K EK$ are infinite dimensional, it makes sense to talk about the normal bundle to the embedding, which is a finite dimensional vector bundle on $X_\beta \times_K EK$.

(My apologies to the reader for the overuse of the letter E. In EK it stands for the universal bundle, in E^* for a cohomology theory, and in E_2 for a term in a spectral sequence. The notation is terrible, but completely standard.)

Kirwan's theorem (Theorem 5.4 of [3]) states that if for the generalised cohomology theory E^* we take $H^*(-, \mathbb{Q})$, that is ordinary cohomology with rational coefficients, then the spectral sequence above degenerates at E_2.

Although Kirwan does not state her theorem in greater generality, the following can be deduced from the same proof.

Theorem 1.2. (proved by the same proof Kirwan gives). *Let all the notation be as above. Then there exists an integer N, depending on X and on the action of K on X, so that if E^* is any complex oriented cohomology theory with coefficients in which N is inverted, then the spectral sequence above degenerates.*

Remark 1.3. The integer N is quite explicit in Kirwan's proof, although it is not clear if the estimate she gives is best possible.

Example 1.4. Many cohomology theories are complex oriented. For example, K–theory, complex cobordism and the Morava K–theories all are.

Problem 1.5. Is it true that the spectral sequence degenerates for all cohomology theories, and not only complex oriented ones?

If the answer to Problem 1.5 turns out to be positive, many consequences follow. In all that follows, let the integer N be inverted. It follows immediately that as a spectrum, $X \times_K EK$ must decompose as the wedge of the T_β's, that is the Thom spaces on the normal bundles $X_\beta \times_K EK$'s. After all, the attaching maps must vanish in the cohomology theories given by the suspension spectra of the T_β's, and hence must vanish.

Therefore, if it turns out that the answer to Problem 1.5 is positive, then one can deduce that the splitting which Kirwan discovered must respect the cohomology operations. Suppose for a second that for E^* we take $H^*(-, \mathbb{Z}/p)$. That is we take ordinary cohomology with coefficiants in the finite field $\mathbb{Z}/p$, where p is not a factor of the integer N above. Kirwan discovered that, as a vector space, $H^*(X \times_K EK, \mathbb{Z}/p)$ decomposes as a direct sum of the cohomology of Thom spaces on the strata, that is shifts of the cohomology of the strata. If the answer to Problem 1.5 is positive, this decomposition would have to respect the structure of modules over the Steenrod algebra.

Perhaps I should elaborate this point a little. $H^*(X \times_K EK, \mathbb{Z}/p)$ is a module over the (mod p) Steenrod algebra. Because the space is startified, its cohomology comes with a natural filtration by submodules over the Steenrod algebra. If the answer to Problem 1.5 is positive, then this filtration would decompose as a direct sum, as modules over the Steenrod algebra.

Similarly for complex cobordism. The complex cobordism of $X \times_K EK$, denoted $MU^*(X \times_K EK)$, has a natural filtration that comes from the stratification of $X \times_K EK$. If the answer to Problem 1.5 is positive, then this filtration would have to split, even as modules over the Landweber–Novikov algebra.

It should be noted that Kirwan's proof depends very heavily on the assumption of complex orientability. Without repeating the entire proof, let us sketch enough of it so we can explain how and where the hypothesis of complex orientability enters.

Sketch of Kirwan's proof. By induction, one quickly reduces to the case where there are only two strata. That is, $Y = X \times_K EK$ can be expressed as a disjoint union

$$Y = U \cup Z$$

where U is open and Z is a closed submanifold. In this case, there is a cofibration sequence

$$U \longrightarrow Y \longrightarrow T$$

where T is the Thom space on the normal bundle of Z in Y. There is a corresponding map $E^*(T) \longrightarrow E^*(Y)$, and it clearly suffices to prove the map injective. To do this, we consider the composite

$$Z \longrightarrow Y \longrightarrow T$$

where $Z \longrightarrow Y$ is the inclusion of Z. There is an induced composite

$$E^*(T) \longrightarrow E^*(Y) \longrightarrow E^*(Z).$$

If we could show this composite injective, then the first map must also be injective, and we are done.

But the composite $Z \longrightarrow T$ is nothing more than than the inclusion of the zero section in the Thom space of the normal bundle of $Z \subset Y$. It turns out that the normal bundle can be given the structure of a complex vector bundle, and since E^* is a complex oriented theory, there is a Thom isomorphism of $E^*(T)$ with (a suspension of) $E^*(Z)$. The inclusion $Z \subset T$ corresponds to a map $E^*(T) \longrightarrow E^*(Z)$, and that map is nothing other than multiplication by the Euler class in the ring $E^*(Z)$. It suffices therefore to prove that the Euler class of the normal bundle to Z is a non–zero–divisor in $E^*(Z)$.

So far, the argument is quite formal, but we have already used once the complex orientability of the theory E^*.

Now the argument becomes less formal. Recall that Y is $X \times_K EK$, and $Z \subset Y$ is $X_\beta \times_K EK$, for some $\beta \in B$. It turns out that there is a subset $C_\beta \subset X_\beta$ which is a deformation retract of X_β, with a deformation retraction respecting the K–action. The problem therefore immediately reduces to C_β. Furthermore, there is a torus $T_\beta \subset K$ and a subset $\{Z_\beta \cap \mu^{-1}(\beta)\} \in C_\beta$, so that if $Stab\beta$ is the centraliser of T_β in K, then

$$C_\beta = K \times_{Stab\beta} \left\{Z_\beta \cap \mu^{-1}(\beta)\right\}.$$

That is, $Z_\beta \cap \mu^{-1}(\beta)$ is a subset of C_β invariant under the action of $Stab\beta$, and the natural map

$$K \times_{Stab\beta} \left\{Z_\beta \cap \mu^{-1}(\beta)\right\} \quad \longrightarrow \quad C_\beta$$

is a homeomorphism. Therefore

$$C_\beta \times_K EK = \left\{Z_\beta \cap \mu^{-1}(\beta)\right\} \times_{Stab\beta} E\,Stab\beta$$

We are given some $Stab\beta$ bundle over $Z_\beta \cap \mu^{-1}(\beta)$, and we need to show that its Euler class on the homotopy quotient $\{Z_\beta \cap \mu^{-1}(\beta)\} \times_{Stab\beta} E\,Stab\beta$ is a non–zero–divisor. We have an inclusion of groups $T_\beta \subset Stab\beta \subset K$, for after all the torus T_β must lie in its own centraliser. Therefore we have a map of homotopy quotients

$$\left\{Z_\beta \cap \mu^{-1}(\beta)\right\} \times_{T_\beta} ET_\beta \quad \longrightarrow \quad \left\{Z_\beta \cap \mu^{-1}(\beta)\right\} \times_{Stab\beta} E\,Stab\beta$$

and it turns out to be sufficient to show that the Euler class in $\{Z_\beta \cap \mu^{-1}(\beta)\} \times_{T_\beta} ET_\beta$ is a non–zero–divisor. It also turns out that the action of T_β on $Z_\beta \cap \mu^{-1}(\beta)$ is trivial. But then

$$\left\{Z_\beta \cap \mu^{-1}(\beta)\right\} \times_{T_\beta} ET_\beta \quad = \quad \left\{Z_\beta \cap \mu^{-1}(\beta)\right\} \times BT_\beta$$

and we have to show that here, some Euler class is a non–zero–divisor. But BT_β is the classifying space of the torus T_β, hence a product of $\mathbb{CP}^\infty$'s. Because the theory is assumed complex oriented, we know $E^*(BT_\beta)$, and it is now straightforward to show that the Euler class is, indeed, a non–zero–divisor.

Remark 1.6. This sketch of Kirwan's proof was included mostly to highlight the essential role played by complex orientability. It enters the proof at two crucial points.

2. The arithmetic version

Let X be a projective variety, G a reductive algebraic group, and $\mathcal{L}$ an ample line bundle on X for which the action of G linearizes. If X is defined over the field of complex numbers $\mathbb{C}$, then out of the above one can construct Kirwan's startification as follows.

Some multiple $\mathcal{L}^n$ of $\mathcal{L}$ embeds X into projective space. Choose a maximal compact subgroup $K \subset G$, and on the vector space $H^0(X, \mathcal{L}^n)$ choose an inner product which is preserved by the action of the compact group K. That is, K acts on $H^0(X, \mathcal{L}^n)$ by unitary transformations. Then the projective space on the vector space $H^0(X, \mathcal{L}^n)$ has a Kähler metric associated to the inner product. The natural embedding of X into this projective space allows us to pull back this Kähler metric, and we obtain on X a K–invariant Kähler metric, whose imaginary part is a K–invariant simplectic form. This form has a moment map, coming from the moment map of $\mathbb{P}^N$.

Thus Kirwan's stratification is defined on X. But in fact, the existence of this stratification in the algebro–geometric case is older. It was first discovered by Kempf [2]. Hesselink proved that if k is a perfect field, and the group G, the variety X, the line bundle $\mathcal{L}$ as well as the action of G on X and $\mathcal{L}$ are all defined over k, then all the strata are rational over the field k.

Aside from technical difficulties in defining EG, which we will now ignore, one then has a stratification

$$X \times_G EG = \bigcup_{\beta \in B} X_\beta \times_G EG.$$

Kirwan's proof still shows that the associated spectral sequence in cohomology degenerates at E_2, if by cohomology we mean for example the l–adic cohomology. In other words, the l–adic cohomology of $X \times_G EG$ is a filtered module, where the subquotients are the l–adic cohomology of $X_\beta \times_G EG$, up to suspension. But now the Galois group $Gal(\overline{k}/k)$ acts on everything, and one can ask

Problem 2.1. Is it true that, as a Galois module, the l–adic cohomology of $X \times_G EG$ is a direct sum of suspensions of the l–adic cohomologies of $X_\beta \times_G EG$?

References

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