

Math 104 Review

1. Find the volume of the following surfaces of revolution:

(a) $\frac{\pi}{2}(e^4 - 1)$

(b) $2\pi(e^2 + 1)$

(c) $2\pi(\pi - 1)$

(d) 2π

(e) This problem is a bit too hard, since one can't easily write down the value where $e^x - 1 = 2 - x$.

(f) This problem is a bit too hard, since one can't easily write down the value where $e^x - 1 = 2 - x$.

(g) 2π

(h) π

(i) 8π

(j) $\frac{2\pi}{e}$

(k) $2\pi \left(1 - \frac{2}{e}\right)$

2. Find the surface area of the following surfaces of revolution:

(a) π

(b) $\frac{\pi}{9}(2\sqrt{2} - 1)$

(c) 49π

(d) $\frac{1}{4} \left[\frac{1}{24}x^8 + \frac{1}{4}x^4 - \frac{1}{4}x^{-4} + \frac{1}{3}\ln x \right]_a^b$ where a and b are the endpoints of the curve.

3. Find the arc length of the following curves

(a) $\ln(\sqrt{2} + 1)$

(b) $\frac{1}{8}(e^4 - e^{-4})$

(c) $\frac{14}{3}$

(d) $\frac{2}{3}(1+b)^{\frac{3}{2}} - \frac{2}{3}$ where b is the endpoint

4. Find the center of mass of the following regions

(a) $M_x = \frac{\pi^2}{144} + \frac{\pi}{4\sqrt{3}} - \frac{1}{2}, M_y = \frac{\sqrt{3}}{16} - \frac{\pi}{48}, M = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

(b) $M_x = \frac{\pi}{2} + \frac{4\pi^3}{3}, M_y = \frac{8\pi^3}{3}, M = 2\pi^2$

(c) $M_x = \frac{1}{2}, M_y = \frac{3}{2}, M = 1$

Math 104 Review

5. Evaluate each of the following integrals, or specify if it diverges:

(a) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$

(b) $\int \frac{1}{x^3 - 1} dx = \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2 + x + 1| - \frac{1}{\sqrt{3}} \arctan\left(\frac{2x+1}{3}\right) + C$

(c) $\int \frac{x}{\sqrt{4-x^2}} dx = -\sqrt{4-x^2} + C$

(d) $\int_0^{\pi} 2x \cos x dx = -4$

(e) $\int \sin x \sin 2x dx = \frac{2}{3} \sin^3 x + C$

(f) $\int \cos^3 x \sin^4 x dx = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$

(g) $\int (\ln x)^2 dx = x(\ln x)^2 - 2x \ln x + 2x + C$

(h) $\int_0^2 \frac{x}{x^2 - 1} dx$ diverges

(i) $\int \frac{\sin x \cos x}{\sin^2 x - 4} dx = \frac{1}{2} \ln(\sin^2 x - 4) + C$

(j) $\int x \sqrt{x^2 + 8} dx = \frac{1}{3} (x^2 + 8)^{\frac{3}{2}} + C$

(k) $\int_0^1 x^4 e^x dx = 9e - 24$

(l) $\int_0^1 \frac{1}{\sqrt{1-x}} dx = 2$

(m) $\int \frac{x^2}{x^4 - 1} dx = \frac{1}{4} \ln(x-1) - \frac{1}{4} \ln(x+1) + \frac{1}{2} \arctan x + C$

(n) $\int e^x \sin 4x dx = \frac{-4}{17} e^x \cos(4x) + \frac{1}{17} e^x \sin(4x) + C$

(o) $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x dx = \frac{1}{3}$

6. Do the following integrals converge or diverge?

(a) $\int_0^{\infty} \frac{\sin^2 x}{1 + e^x} dx$ converges

(b) $\int_2^{\infty} \frac{x^2 e^x}{\ln x} dx$ diverges

(c) $\int_1^{\infty} e^{x^2+x+1} dx$ diverges

Math 104 Review

- (d) $\int_1^\infty \frac{x^2 - 2}{x^4 + 3} dx$ converges
7. Determine the constant c so that the following functions are probability density functions. Then, compute the mean and the median.
- (a) $p(x) = c(2 - x)$, $x \in [0, 4]$ can't be a pdf since it will be negative somewhere
- (b) $p(x) = \frac{c}{x^2 + 1}$, $x \in (-\infty, \infty)$ $c = \frac{1}{\pi}$, μ is undefined, $m = 0$
- (c) $f(x) = c \sin^2 x$, $x \in [0, \pi]$ $c = \frac{2}{\pi}$, $\mu = \frac{\pi}{2}$, $m = \frac{\pi}{2}$
8. Find the solution of the following differential equations:
- (a) $y' = 1 - \frac{y}{x}$, $y(2) = -1$ $y(x) = \frac{x}{2} - \frac{4}{x}$
- (b) $(1 + x)y' + y = \sqrt{x}$, $y(0) = 1$ $y(x) = \frac{2x^{\frac{3}{2}} + 3}{3x + 3}$
- (c) $y' + 2y = 3$ $y(x) = Ce^{-2x} + \frac{3}{2}$
- (d) $y' = \frac{\tan y}{x}$, $y(1) = \frac{\pi}{2}$ $y(x) = \arcsin(e^{x-1})$
- (e) $e^{2x}y' + e^{2x}y = 2x$ $y(x) = -2xe^{-2x} - 2e^{-2x} + Ce^{-x}$
- (f) $xy' + 3y = \frac{\sin x}{x^2}$ $y(x) = -x^{-3} \cos x + Cx^{-3}$
9. Do the following sequences converge or diverge? Compute the limits of the convergent sequences.
- (a) $\{(n^2 + n)^{\frac{1}{n}}\}$ 1
- (b) $\left\{\arctan\left(\frac{n^2}{1 + n^2}\right)\right\}$ $\frac{\pi}{4}$
- (c) $\left\{\cos\left(\frac{\sqrt{n}}{1 + n}\right)\right\}$ 1
- (d) $\left\{n^2\left(1 - \cos\left(\frac{1}{n}\right)\right)\right\}$ $\frac{1}{2}$
- (e) $\{\sin(\arctan(\ln(n)))\}$ 1
10. For each of the following series, say whether it converges (absolutely or conditionally) or diverges, and explain why.
- (a) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ absolutely converges (ratio test)

Math 104 Review

(b) $\sum_{n=1}^{\infty} n^3 e^{-n^2}$	absolutely converges (ratio test)
(c) $\sum_{n=1}^{\infty} \frac{4+n}{3+2n}$	diverges (n^{th} term test)
(d) $\sum_{n=1}^{\infty} \frac{n^2+5}{\sqrt{n^5+2}}$	diverges (limit comparison test)
(e) $\sum_{n=1}^{\infty} \frac{e^n}{(2+\frac{1}{n})^n}$	diverges (root test)
(f) $\sum_{n=1}^{\infty} \left(\frac{-4}{5}\right)^n$	absolutely converges (geometric series)
(g) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^3+1}}$	diverges (n^{th} term test)
(h) $\sum_{n=1}^{\infty} \frac{3^n}{n}$	diverges (n^{th} term test)
(i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$	absolutely converges (ratio test)
(j) $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdots (2n-1)}$	absolutely converges (ratio test)
(k) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$	diverges (limit comparison test)
(l) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$	conditionally converges (alternating harmonic series)
(m) $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$	absolutely converges (limit comparison test)
(n) $\sum_{n=1}^{\infty} \frac{n}{2^n}$	absolutely converges (ratio test)
(o) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$	diverges (n^{th} term test)
(p) $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n}}$	absolutely converges (limit comparison test)
(q) $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$	absolutely converges (ratio test)
(r) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$	diverges (integral test)

11. For each of the following Taylor series, determine the precise interval of convergence.

- (a) $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2 4^n}$ $[\frac{1}{2}, \frac{9}{2}]$
- (b) $\sum_{n=1}^{\infty} \frac{(x-4)^n}{5^n}$ $(-1, 9)$
- (c) $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n 3^n}$ $(-1, 5]$
- (d) $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n^2}$ $[\frac{5}{3}, \frac{7}{3}]$
- (e) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$ $[0, 2]$
- (f) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$ $(-2, 2)$

12. Write the second Taylor polynomial for \sqrt{x} , centered at 25. Use this polynomial to estimate $\sqrt{26}$. Also, give an estimate of the error.

The second Taylor polynomial is $P_2(x) = 5 + \frac{1}{10}(x-5) - \frac{1}{1000}(x-5)^2$.
 We therefore have an estimate of $\sqrt{26} \approx P_2(26) = \frac{5099}{1000}$.
 The error of this estimate is within $\frac{1}{6 \cdot 5^5} = \frac{1}{18750}$.

13. Use the first two non-zero terms of an appropriate series to give an approximation of

$$\int_0^1 \sin(x^2) dx.$$

Give an estimate of the error.

Let $F(x) = \int \sin(x^2) dx$ be the antiderivative of $\sin(x^2)$ with $F(0) = 0$.

The 10th Taylor polynomial for $F(x)$ is $P_7(x) = \frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7$. We therefore have an estimation that $F(1) \approx P_7(1) = \frac{13}{42}$.

The error for this estimate is $\frac{|F^{(11)}(1)|}{11!}$. This error term is somewhat difficult to compute since it takes a while to differentiate $F(x)$. It turns out that the error is bounded by $\frac{223}{2494800}$.

14. What is the sixth Taylor polynomial for xe^{3x} , centered at 0?

The 6th Taylor polynomial is $x + 3x^2 + \frac{3^2}{2!}x^3 + \frac{3^3}{3!}x^4 + \frac{3^4}{4!}x^5 + \frac{3^5}{5!}x^6$

15. What is $\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{e^{x^2} - 1}$?

Math 104 Review

16. Which of the following is the closest value of

$$\int_0^{0.1} \cos(x^2) dx?$$

Justify your answer.

- (A) 0.0998 (B) 0.0999 (C) 0.1000 (D) 0.1001 (E) 0.1002 (F) 20.1003

The 1st Taylor polynomial for $\int \cos(x^2) dx$ is $P_1(x) = x$ (here, I'm choosing the antiderivative that evaluates to 0 at 0).

We therefore have an estimation $\int_0^{0.1} \cos(x^2) dx \approx (0.1)$.

The error for $P_1(x)$ is $\frac{|2x \sin(x^2)|}{2!} |x|^2$. If we plug in 0.1, we can bound this error by 0.00001. The answer is therefore (C).

17. Which of the following is the closest value of $\cos(\frac{\pi}{5})$?

- (A) $\frac{198}{200}$ (B) $\frac{199}{200}$ (C) 2 (D) $\frac{201}{200}$ (E) $\frac{202}{200}$ (F) $\frac{203}{200}$

Let's approximate $\cos(\frac{\pi}{5})$ using the 3rd Taylor polynomial of $\cos x$, which is $P_3(x) = 1 - \frac{x^2}{2}$. Since $3 \leq \pi \leq 4$, we have that $\frac{132}{200} \leq P_3(\frac{\pi}{5}) \leq \frac{164}{200}$. It therefore appears that the best answer is (A).

Indeed, the error for $P_3(\frac{\pi}{5})$ is $\frac{|\cos(\frac{\pi}{5})|}{4!} |\frac{\pi}{5}|^4 \leq \frac{4^4}{4!5^4} = \frac{32}{1875}$