Final Exam

Math 180, Spring 2011

You have 120 minutes.

No books, phones, calculators, talking, etc. You are allowed a single hand-written 4×6 inch note card (both sides), but **no other materials**. You may write on the back side of the sheets if necessary.

Please make your answers as clear and complete as possible. When asked to explain answers, **please write in complete sentences**. In some cases you will be instructed to leave your numerical answers written in a form that could be entered in a calculator.

Good luck!

Name: _____

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination."

Signature: _____

Total Score: _____ (/66 points)

Formulas from Probability and Statistics

Impossible event: if $\emptyset$ is the impossible event, then

$$P(\emptyset) = 0$$

Certain event: if S is the event consisting of all outcomes, then

$$P(S) = 1$$

Complement rule: if A is an event and A^c is the complementary event, then

$$P(A) + P(A^c) = 1$$

Subset rule: if every outcome in A is also in B, then

$$P(A) \leq P(B)$$

Addition rule: if $E_1, E_2, \ldots E_n$ are mutually exclusive events, and if $E = E_1 \cup E_2 \cup \ldots \cup E_n$,

$$P(E) = P(E_1) + P(E_2) + \ldots + P(E_n)$$

Inclusion-exclusion rule: if A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Independent events: if A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

Conditional probability definiton:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication rule: if A and B are any two events, then

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Bayes' formula:

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(A^c) \cdot P(B|A^c)}$$

Expected value, variance: if there are n outcomes with payoffs equal to $x_1, x_2, \ldots, x_n$, occurring with probabilitites $p_1, p_2, \ldots, p_n$, then the expected payoff is:

$$\mu = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n$$

The variance of the payoff is:

$$\sigma^2 = (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \ldots + (x_n - \mu)^2 p_n$$

Bernoulli trials: In n repetitions of a Bernoulli trial with probability of success equal to p, the expected number of successes is:

$$\mu = np.$$

The variance of the number of successes is:

$$\sigma^2 = np(1-p)$$

z-scores: For a normally-distributed variable with mean μ and standard deviation σ, the z-score of the value x is

$$z = \frac{x - \mu}{\sigma}$$

Interquartile range: The IQR is $Q_3 - Q_1$, where Q_1 is the median of the first and second quartiles, and Q_3 is the median of the third and fourth quartiles.

Survey with a yes/no question: If you question n people and x of them answer "yes", then $\hat{p} = \frac{x}{n}$. $\hat{p}$ follows a normal distribution with mean equal to p (the true proportion of people who would answer "yes") and standard deviation:

$$\sigma = \frac{\sqrt{p(1-p)}}{\sqrt{n}}$$

1. *(10 points)* We play a game in which I simultaneously flip a coin and roll a single die (with sides numbered 1–6). If the coin shows heads, you pay me (in dollars) the number shown on the die. If the coin shows tails, I pay you (in dollars) the number shown on the die.

   (a) How many outcomes are there?

   (b) Let O be the event that an odd number is shown on the die, and H be the event that the coin shows heads. Are O and H mutually exclusive? Why or why not?

   (c) Are O and H independent? Why or why not?

   (d) Find the probability of O, given H.

   (e) What is *your* expected payout for this game? Calculate or explain.

2. *(4 points)* Explain in your own words the *defense attorney's fallacy*, and why it is an invalid form a reasoning. You may wish to refer to the O.J. Simpson trial in your answer.

3. *(6 points)* A blood test for a rare genetic disorder (affecting only 1 out of 250,000 people) correctly identifies the disease in 96% of affected patients, with a false positive rate of 2%. *If necessary, leave your answers in terms of an expression that could be entered into a calculator.*

   (a) If you test positive, how likely is it that you actually have the disease?

   (b) If you have the disease, how likely is it that you would test positive?

4. *(6 points)* Suppose you flip a fair coin 100 times.

   (c) What is the probability of getting between 45 and 50 heads? (Use the normal distribution to give an approximate answer.)

   (b) What is the probability of getting 65 or more heads? (Use the normal distribution to give an approximate answer.)

   (c) What is the probability of getting at least 99 heads? (*Leave your answer as an exact expression* that could be entered in a calculator.)

5. *(8 points)* Suppose you want to know how many Philadelphia residents actually understand what "margin-of-error" means in surveys. You get responses from 150 people who walk by the corner of Walnut and 34th, and 42% answer correctly. You use these results to estimate the true proportion of residents who would answer correctly.

   (a) Find an expression for the margin of error at the **98**% confidence level. *Leave your answer as an expression that could be entered into a calculator.*

   (b) *How many more* people would you need to survey to cut your margin of error in half?

   (c) Aside from surveying more students, what is another valid way to report a smaller margin of error?

   (d) Identify any likely sources of bias in this survey.

6. *(6 points)* Two people play a zero sum "matching pennies" game as follows. On the count of three, they each hold out a penny, showing either heads or tails. (They each get to choose which side of the coin is showing.) Below is a payoff matrix, where a positive number indicates a gain for the row player, and a negative number indicates a loss for the row player. (The first row and first column correspond to heads.)

$$\begin{bmatrix} 2 & -1 \\ -5 & 4 \end{bmatrix}$$

(a) Determine the optimal strategy for the row player, and calculate the value of the game.

(b) Would you rather play this game as the row player or column player? Why?

7. *(8 points)* Two warring armies have reached a tentative cease-fire agreement. Each army has the option of breaking the agreement (B) or honoring the agreement (H). If both break the agreement, they fight, and both sides lose some soldiers. Assign this outcome a value of -1 for each side. If both honor the agreement, no fighting occurs; assign this the value 0 for each side. If one side breaks the agreement and the other honors the agreement, the side that broke the agreement gets a +5 advantage, and the side that honored the agreement gets -10, since they are unprepared for the fight.

   (a) Draw the game table.

   (b) Is this game zero sum? Explain.

   (c) Identify any Nash equilibria.

   (d) Of the famous games we discussed, which does this most resemble, and why?

8. *(6 points)* Suppose you own a restaurant and are considering purchasing a liquor license. You estimate that serving alcohol would earn you an additional $40,000 profit per year into the forseeable future. (For simplicity, assume the $40,000 profit arrives at the start of each year, beginning today.) Assuming an interest rate of 10%. What is the most you would be willing to pay for the license? (*Simplify your answer as much as possible, but you may leave it in a form that could be entered on a calculator.*)

9. *(4 points)* If you invest \$1,000 today at a high interest rate of 12%, about how many years are required for your investment to reach \$8,000 in value?

10. *(8 points)* Consider an election for class president with 100 voters and 3 candidates: A, B, and C. The results are:

    - 39 votes for ABC (meaning A is preferred the most, followed by B, followed by C).
    - 35 votes for BCA
    - 26 votes for CAB

    (a) Determine the election winner using the instant run-off method.

    (b) In the next election, the same candidates run, and the same 100 students vote. However, 10 of the students who voted BCA before *change their vote* to ABC (and everyone else stays the same). Who wins the election now (with instant run-off)?

    (c) In your own words, what do your answers to (a) and (b) demonstrate about the instant run-off system of voting?

    (d) Is there a Condorcet winner in the original election? If so, who? If none, why not?

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

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Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997