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- Let A_k be the event that you roll k. What is $P(A_k)$?

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  - In ten tosses, there are $\underbrace{2 \times 2 \times \ldots \times 2}_{\text{ten times}} = 2^{10} = 1024$ possible outcomes.

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- By the complement rule, $P(A) = 1 - P(A^c) = 1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.999$.

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  - B^c is the event that T shows up at most once, so

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- Observe A^c is the event that all n people have *distinct* birthdays. Find $P(A^c)$ first.

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- When 23 people are chosen randomly, there is a better than 50% chance that at least two of them share a birthday!

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- Daily Show interviewee: the chance of the LHC destroying the Earth is 50%, since it will either happen or it won't.

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