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  - Note that the expected payout is a weighted average of payouts

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- A best response to a strategy will consist of pure strategies that have the **same** (high) expected payout

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- Note that pure Nash equilibria are still Nash equilibria

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- Note that the expected payoff for each player is 0 (the game is fair)

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Theorem (Nash)

Suppose that:

- a game has finitely many players
- each player has finitely many pure strategies
- we allow for mixed strategies

Then the game admits a Nash equilibrium

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  - Strategies in Nash equilibrium are $(.7, .3)(p = .7)$ and $(.6, .4)(q = .6)$

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- Percent of population with mutation will grow until Nash equilibrium is reached