

Inheritance

- ▶ Two parties are fighting over inheritance (say \$100)

Inheritance

- ▶ Two parties are fighting over inheritance (say \$100)
 - ▶ Party *A* claims \$100
 - ▶ Party *B* claims \$50

Inheritance

- ▶ Two parties are fighting over inheritance (say \$100)
 - ▶ Party *A* claims \$100
 - ▶ Party *B* claims \$50
- ▶ How should one split the \$100?

Two Methods

Two Methods:

- ▶ Equal division

Two Methods

Two Methods:

- ▶ Equal division
 - ▶ Both parties get \$50

Two Methods

Two Methods:

- ▶ Equal division
 - ▶ Both parties get \$50
 - ▶ All parties get the same amount

Two Methods

Two Methods:

- ▶ Equal division
 - ▶ Both parties get \$50
 - ▶ All parties get the same amount
- ▶ Proportional Division

Two Methods

Two Methods:

- ▶ Equal division
 - ▶ Both parties get \$50
 - ▶ All parties get the same amount
- ▶ Proportional Division
 - ▶ Party *A* gets \$67
 - ▶ Party *B* gets \$33

Two Methods

Two Methods:

- ▶ Equal division
 - ▶ Both parties get \$50
 - ▶ All parties get the same amount
- ▶ Proportional Division
 - ▶ Party *A* gets \$67
 - ▶ Party *B* gets \$33
 - ▶ Parties get the proportion of what they claimed to the sums of all claims

Two Methods

Two Methods:

- ▶ Equal division
 - ▶ Both parties get \$50
 - ▶ All parties get the same amount
- ▶ Proportional Division
 - ▶ Party *A* gets \$67
 - ▶ Party *B* gets \$33
 - ▶ Parties get the proportion of what they claimed to the sums of all claims
- ▶ Resolution depends on social customs

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people
- ▶ Goods can be:

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people
- ▶ Goods can be:
 - ▶ divisible: can split goods into any proportions (money)

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people
- ▶ Goods can be:
 - ▶ divisible: can split goods into any proportions (money)
 - ▶ indivisible: cannot split goods into any proportions ({piano, car, dog})

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people
- ▶ Goods can be:
 - ▶ divisible: can split goods into any proportions (money)
 - ▶ indivisible: cannot split goods into any proportions ({piano, car, dog})
- ▶ Not all people have to value the same goods the same way

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people
- ▶ Goods can be:
 - ▶ divisible: can split goods into any proportions (money)
 - ▶ indivisible: cannot split goods into any proportions ({piano, car, dog})
- ▶ Not all people have to value the same goods the same way
- ▶ People may have different levels of entitlement

Fair Division

This is an example of a **fair division** problem:

- ▶ Want to split some goods *fairly* among some people
- ▶ Goods can be:
 - ▶ divisible: can split goods into any proportions (money)
 - ▶ indivisible: cannot split goods into any proportions ({piano, car, dog})
- ▶ Not all people have to value the same goods the same way
- ▶ People may have different levels of entitlement
- ▶ Can divide inheritance, chores, business profits, Berlin, cake, ...

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:
 - ▶ simple fair division: $u_i(X_i) \geq \frac{1}{n}$

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:
 - ▶ simple fair division: $u_i(X_i) \geq \frac{1}{n}$
 - ▶ envy-free: $u_i(X_i) \geq u_i(X_j)$

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:
 - ▶ simple fair division: $u_i(X_i) \geq \frac{1}{n}$
 - ▶ envy-free: $u_i(X_i) \geq u_i(X_j)$
 - ▶ equitable: $u_i(X_i) = u_j(X_j)$

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:
 - ▶ simple fair division: $u_i(X_i) \geq \frac{1}{n}$
 - ▶ envy-free: $u_i(X_i) \geq u_i(X_j)$
 - ▶ equitable: $u_i(X_i) = u_j(X_j)$
 - ▶ Pareto optimal: no other division would make someone else better off without making someone else worse off

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:
 - ▶ simple fair division: $u_i(X_i) \geq \frac{1}{n}$
 - ▶ envy-free: $u_i(X_i) \geq u_i(X_j)$
 - ▶ equitable: $u_i(X_i) = u_j(X_j)$
 - ▶ Pareto optimal: no other division would make someone else better off without making someone else worse off
 - ▶ Problem:

Fair Division

- ▶ Let $u_i(X_j)$ be the value that person i assigns to j 's division
- ▶ Fairness can be measured by:
 - ▶ simple fair division: $u_i(X_i) \geq \frac{1}{n}$
 - ▶ envy-free: $u_i(X_i) \geq u_i(X_j)$
 - ▶ equitable: $u_i(X_i) = u_j(X_j)$
 - ▶ Pareto optimal: no other division would make someone else better off without making someone else worse off
 - ▶ Problem: Someone getting everything, and everyone else getting nothing, is Pareto optimal

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)
- ▶ Problem: estate is not worth 600

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)
- ▶ Problem: estate is not worth 600
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$ (this agrees with equal division)

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)
- ▶ Problem: estate is not worth 600
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$ (this agrees with equal division)
 - ▶ If estate worth is 300, the wives receive 50, 100, 150 (this agrees with proportional division)

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)
- ▶ Problem: estate is not worth 600
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$ (this agrees with equal division)
 - ▶ If estate worth is 300, the wives receive 50, 100, 150 (this agrees with proportional division)
 - ▶ If estate worth is 200, the wives receive 50, 75, 75 (???)

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)
- ▶ Problem: estate is not worth 600
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$ (this agrees with equal division)
 - ▶ If estate worth is 300, the wives receive 50, 100, 150 (this agrees with proportional division)
 - ▶ If estate worth is 200, the wives receive 50, 75, 75 (???)
- ▶ Is there a coherent rule that outlines these cases?

Three Wives

A rule in the Talmud:

- ▶ Man is married to three women
- ▶ Upon husband's death, each wife is to receive 100, 200, 300 (zuz)
- ▶ Problem: estate is not worth 600
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$ (this agrees with equal division)
 - ▶ If estate worth is 300, the wives receive 50, 100, 150 (this agrees with proportional division)
 - ▶ If estate worth is 200, the wives receive 50, 75, 75 (???)
- ▶ Is there a coherent rule that outlines these cases?
 - ▶ Solved by game theorists in 1985

Equal Division of Contested Sums

- ▶ Trying to fairly divide amongst two people

Equal Division of Contested Sums

- ▶ Trying to fairly divide amongst two people
- ▶ Idea:
 - ▶ Give everyone their uncontested amounts

Equal Division of Contested Sums

- ▶ Trying to fairly divide amongst two people
- ▶ Idea:
 - ▶ Give everyone their uncontested amounts
 - ▶ Split contested amount in half

Equal Division of Contested Sums

- ▶ Two parties are trying to split 100

Equal Division of Contested Sums

- ▶ Two parties are trying to split 100
- ▶ Party A claims 100

Equal Division of Contested Sums

- ▶ Two parties are trying to split 100
- ▶ Party *A* claims 100
- ▶ Party *B* claims 50

Equal Division of Contested Sums

- ▶ Two parties are trying to split 100
- ▶ Party *A* claims 100
- ▶ Party *B* claims 50
- ▶ Using equal division of contested sums, how much does each party receive?

Equal Division of Contested Sums

- ▶ Two parties are trying to split 100
- ▶ Party *A* claims 100
- ▶ Party *B* claims 50
- ▶ Using equal division of contested sums, how much does each party receive?
 - ▶ *A* receives 75
 - ▶ *B* receives 25

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party *A* claims 100

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party *A* claims 100
- ▶ Party *B* claims 300

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party A claims 100
- ▶ Party B claims 300
- ▶ If estate is worth 80, how much does each party receive?

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party *A* claims 100
- ▶ Party *B* claims 300
- ▶ If estate is worth 80, how much does each party receive?
 - ▶ *A* receives 40
 - ▶ *B* receives 40

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party A claims 100
- ▶ Party B claims 300
- ▶ If estate is worth 80, how much does each party receive?
 - ▶ A receives 40
 - ▶ B receives 40
- ▶ If estate is worth 125, how much does each party receive?

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party *A* claims 100
- ▶ Party *B* claims 300
- ▶ If estate is worth 80, how much does each party receive?
 - ▶ *A* receives 40
 - ▶ *B* receives 40
- ▶ If estate is worth 125, how much does each party receive?
 - ▶ *A* receives 50
 - ▶ *B* receives 75

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party *A* claims 100
- ▶ Party *B* claims 300
- ▶ If estate is worth 80, how much does each party receive?
 - ▶ *A* receives 40
 - ▶ *B* receives 40
- ▶ If estate is worth 125, how much does each party receive?
 - ▶ *A* receives 50
 - ▶ *B* receives 75
- ▶ If estate is worth 200, how much does each party receive?

Equal Division of Contested Sums

- ▶ Two parties are trying to split inheritance
- ▶ Party *A* claims 100
- ▶ Party *B* claims 300
- ▶ If estate is worth 80, how much does each party receive?
 - ▶ *A* receives 40
 - ▶ *B* receives 40
- ▶ If estate is worth 125, how much does each party receive?
 - ▶ *A* receives 50
 - ▶ *B* receives 75
- ▶ If estate is worth 200, how much does each party receive?
 - ▶ *A* receives 50
 - ▶ *B* receives 150

Three Wives

- ▶ Back to rule in the Talmud:

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$
 - ▶ If estate worth is 300, the wives receive 50, 100, 150

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$
 - ▶ If estate worth is 300, the wives receive 50, 100, 150
 - ▶ If estate worth is 200, the wives receive 50, 75, 75

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$
 - ▶ If estate worth is 300, the wives receive 50, 100, 150
 - ▶ If estate worth is 200, the wives receive 50, 75, 75

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$
 - ▶ If estate worth is 300, the wives receive 50, 100, 150
 - ▶ If estate worth is 200, the wives receive 50, 75, 75
- ▶ The allotment of any two wives is split using the above rule!
(three things to check in each case)

Three Wives

- ▶ Back to rule in the Talmud:
- ▶ Wives claim 100, 200, 300
 - ▶ If estate worth is 100, each wife receives $33\frac{1}{3}$
 - ▶ If estate worth is 300, the wives receive 50, 100, 150
 - ▶ If estate worth is 200, the wives receive 50, 75, 75
- ▶ The allotment of any two wives is split using the above rule!
(three things to check in each case)
- ▶ Need method for when there are more than two parties

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)
- ▶ Divide estate equally until 1 receives half of their claim

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)
- ▶ Divide estate equally until 1 receives half of their claim
- ▶ Divide estate equally among 2 through n until 2 receives half of their claim

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)
- ▶ Divide estate equally until 1 receives half of their claim
- ▶ Divide estate equally among 2 through n until 2 receives half of their claim
- ▶ Proceed until everyone has half of their claim

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)
- ▶ Divide estate equally until 1 receives half of their claim
- ▶ Divide estate equally among 2 through n until 2 receives half of their claim
- ▶ Proceed until everyone has half of their claim
- ▶ Give n money until their loss equals $n - 1$'s loss (loss is money owed minus money given)

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)
- ▶ Divide estate equally until 1 receives half of their claim
- ▶ Divide estate equally among 2 through n until 2 receives half of their claim
- ▶ Proceed until everyone has half of their claim
- ▶ Give n money until their loss equals $n - 1$'s loss (loss is money owed minus money given)
- ▶ Give $n - 1$ and n money until their loss equals $n - 2$'s loss

Algorithm

Want algorithm for the equal division of contested sums between n claims

- ▶ Order claims from lowest to highest (from 1 to n)
- ▶ Divide estate equally until 1 receives half of their claim
- ▶ Divide estate equally among 2 through n until 2 receives half of their claim
- ▶ Proceed until everyone has half of their claim
- ▶ Give n money until their loss equals $n - 1$'s loss (loss is money owed minus money given)
- ▶ Give $n - 1$ and n money until their loss equals $n - 2$'s loss
- ▶ Proceed until all losses are equal

Equal Division of Contested Sums

- ▶ Suppose the estate is worth 550

Equal Division of Contested Sums

- ▶ Suppose the estate is worth 550
- ▶ How much do the parties receive using the algorithm?

Equal Division of Contested Sums

- ▶ Suppose the estate is worth 550
- ▶ How much do the parties receive using the algorithm?
 - ▶ $83\frac{1}{3}$, $183\frac{1}{3}$, and $283\frac{1}{3}$