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- Goods can be divisible or indivisible
- Privacy: splitting should occur without parties knowing other parties' preferences

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- This problem is similar to splitting assets of failed partnerships, ...

Pizza Theorem

A similar example: splitting a pizza

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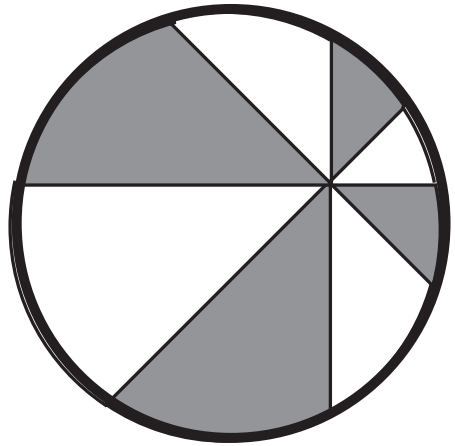

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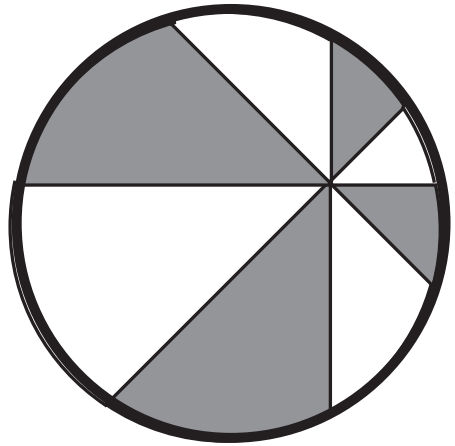

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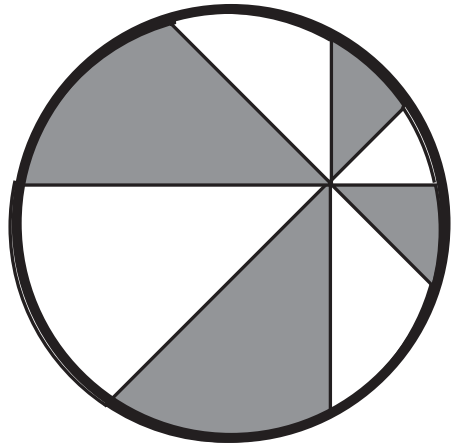

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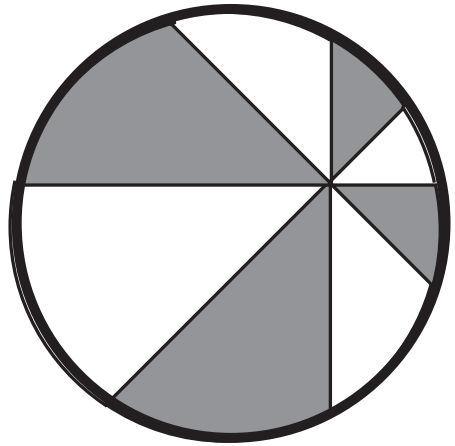

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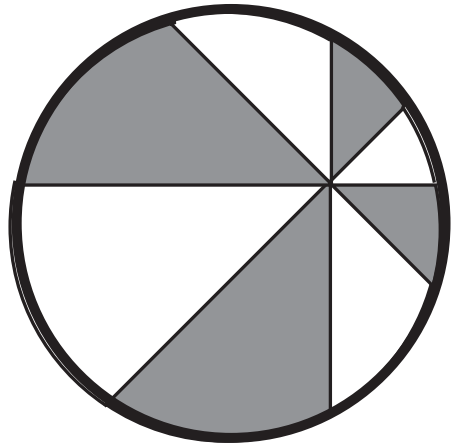

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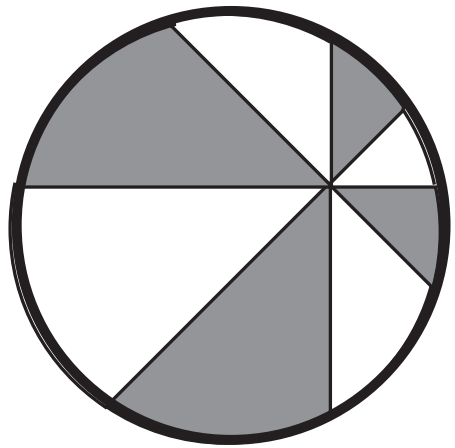

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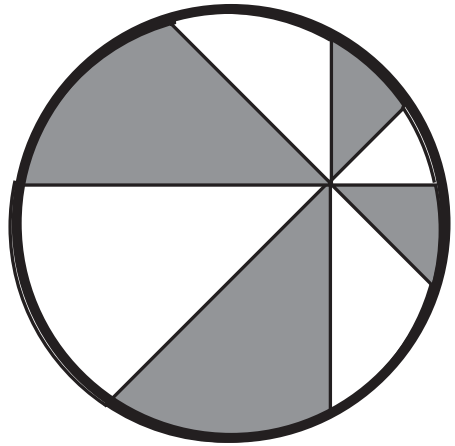

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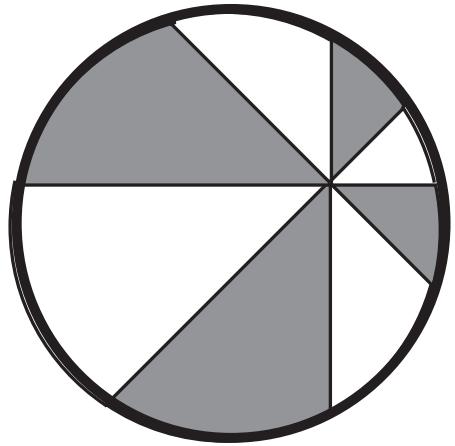

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- The area of the grey slice equals the area of white slices

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  - Doesn't allow parties to act rationally (timed event)

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- Another method (lone divider method):

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- Albert, Beatrice and Clyde are trying to divide a cake

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- Another method (lone divider method):
- Albert, Beatrice and Clyde are trying to divide a cake
- Step 1: Albert cuts the cake into three pieces p_1, p_2, p_3

Cake Cutting

- Step 2: Albert, Beatrice and Clyde (privately) bid on their valuation of each piece:

	Shares		
	p_1	p_2	p_3
Albert	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Beatrice	20%	40%	40%
Clyde	40%	30%	30%

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- Rows will add up to 100%
- Albert's row should be $33\frac{1}{3}\%$
- Everybody has some piece of value $\geq 33\frac{1}{3}\%$

Cake Cutting

- Step 3: Each player declares which pieces they think are fair

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- Step 4: Give each player a piece that they want
  - Albert: p_2, Beatrice: p_3, Clyde: p_1 OR
  - Albert: p_3, Beatrice: p_2, Clyde: p_1

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- Step 6: Use divider-chooser method

Lone Divider Method

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  - It's inefficient