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  - Bank unwilling to lend you money

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- This is referred to as **risk-neutral pricing**

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  - The initial value of the portfolio is

$$\$\frac{0.6 \cdot 160.53 + 0.4 \cdot 50}{1.01} = \$115.17$$

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  - So $x = 500$
- Since we know x and $P = x - y$, we also know y

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- Using risk-neutral pricing, the price of the portfolio will be

$$\$\frac{0.36 \cdot 160.53 + 0.48 \cdot 50 + 0.16 \cdot 0}{(1.02)^2} = \$78.61$$

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    - Merton and Scholes won the 1997 Nobel Prize in Economics for this model