Appendix B

Basic Analysis

This appendix contains some of the basic definitions and results from analysis that are used in this book. Many good treatments of this material are available; for example, [27], [111], or [121].

B.1 Sequences

A *sequence* is an ordered list of objects. As such it can be described by a function defined on a subset of the integers of the form $\{M, M+1, \ldots, N-1, N\}$. This set is called the *index set*. If both M and N are finite, then the sequence is finite. If $M = -\infty$ and N is finite or M is finite and $N = \infty$, then it is an infinite sequence. If the index set equals $\mathbb{Z}$, then the sequence is *bi-infinite*. In this section and throughout most of the book, the term *sequence* usually refers to an infinite sequence. In this case the index set is usually taken to be the positive integers $\mathbb{N}$. For example, a sequence of real numbers is specified by a function

$$x : \mathbb{N} \longrightarrow \mathbb{R}.$$

With this notation the nth term of the sequence would be denoted $x(n)$. It is *not* customary to use functional notation but rather to use subscripts to label the terms of a sequence. The nth term is denoted by x_n and the totality of the sequence by $< x_n >$. This distinguishes a sequence from the *unordered* set consisting of its elements, which is denoted $\{x_n\}$.

Given a sequence $< x_n >$, a *subsequence* is defined by selecting a subset of $\{x_n\}$ and keeping them in the same order as they appear in $< x_n >$. In practice, this amounts to defining a monotone increasing function from $\mathbb{N}$ to itself. We denote the value of this function at j by n_j. In order to be monotone increasing, $n_j < n_{j+1}$, for every j in $\mathbb{N}$. The jth term of the subsequence is denoted by x_{n_j}, and the totality of the subsequence by $< x_{n_j} >$. As an example, consider the sequence $x_n = (-1)^n n$; setting $n_j = 2j$ defines the subsequence $x_{n_j} = (-1)^{2j} 2j$.

***Definition* B.1.1.** A sequence of real numbers, $< x_n >$, has a *limit* if there is a number L such that, given any $\epsilon > 0$, there exists an integer $N > 0$, such that

$$|x_n - L| < \epsilon \qquad \text{whenever } n > N.$$

A sequence with a limit is called a *convergent sequence*: we then write

$$\lim_{n\to\infty} x_n = L.$$

When a limit exists it is unique. A sequence may fail to have limit, but it may have a subsequence that converges. In this case the sequence is said to have a *convergent subsequence*. For example, $x_n = (-1)^n$ is not convergent, but the subsequence defined by $n_j = 2j$ is.

The computation of limits is facilitated by the rules for computing limits of algebraic combinations of convergent sequences.

Theorem B.1.1 (Algebraic rules for limits). *Suppose that $< x_n >, < y_n >$ are convergent sequences of real numbers. Then*

$$\begin{aligned}
\lim_{n\to\infty} ax_n &\textit{ exists and equals } a\lim_{n\to\infty} x_n, \textit{ for all } a \in \mathbb{R},\\
\lim_{n\to\infty}(x_n + y_n) &\textit{ exists and equals } \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,\\
\lim_{n\to\infty}(x_n y_n) &\textit{ exists and equals } (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n),\\
\textit{provided } \lim_{n\to\infty} y_n \neq 0, \textit{ then } \lim_{n\to\infty}\frac{x_n}{y_n} &\textit{ exists and equals } \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.
\end{aligned} \tag{B.1}$$

In this theorem the nontrivial claim is that the limits exist; once this is established it is easy to show what the limits must be.

A problem of fundamental importance is to decide whether or not a sequence has a limit. A sequence $< x_n >$ is bounded if there is a number M such that

$$|x_n| < M \qquad \text{for all } n.$$

A sequence is non-increasing if

$$x_n \geq x_{n+1} \qquad \text{for all } n,$$

and nondecreasing if

$$x_n \leq x_{n+1} \qquad \text{for all } n.$$

The *completeness axiom* of the real numbers states that *a bounded non-increasing or non-decreasing sequence of real numbers has a limit.* If a bounded sequence is neither non-decreasing nor non-increasing, then the only general theorem about convergence is the Bolzano-Weierstrass theorem.

Theorem B.1.2 (Bolzano-Weierstrass theorem). *A bounded sequence of real numbers has a convergent subsequence.*

Note that this theorem does *not* assert that any bounded sequence converges but only that any bounded sequence has a convergent subsequence.

The following two lemmas are very useful for studying limits of sequences.

Lemma B.1.1 (Squeeze lemma). *If* $< x_n >, < y_n >, < z_n >$ *are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n$$

and x_n *and* z_n *are convergent with*

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n,$$

then y_n *converges with*

$$\lim_{n\to\infty} y_n = L.$$

Lemma B.1.2. *If* $x_n \geq 0$ *for* n *in* $\mathbb{N}$ *and* $< x_n >$ *is a convergent sequence, then*

$$\lim_{n\to\infty} x_n \geq 0.$$

In the preceding discussion it is assumed that the limit is known in advance. There is a criterion, due to Cauchy, that states that a given sequence has a limit but makes no reference to the limit itself.

Theorem B.1.3 (Cauchy criterion for sequences). *If* $< x_n >$ *is a sequence of real numbers such that, given* $\epsilon > 0$ *there exists an* N *for which*

$$|x_n - x_m| < \epsilon \qquad \text{whenever both } n \text{ and } m \text{ are greater than } N,$$

then the sequence is convergent.

A sequence satisfying this condition is called a *Cauchy sequence*.

B.2 Series

A series is the sum of an infinite sequence; it is denoted by

$$\sum_{n=1}^{\infty} x_n.$$

***Definition* B.2.1.** A *series* converges if the *sequence* of partial sums

$$s_k = \sum_{n=1}^{k} x_n,$$

converges. In this case

$$\sum_{n=1}^{\infty} x_n \stackrel{d}{=} \lim_{k\to\infty} s_k.$$

If a series does not converge ,then it diverges.

***Definition* B.2.2.** A series converges *absolutely* if the sum of the absolute values

$$\sum_{n=1}^{\infty} |x_n|$$

converges.

The following theorem lists the elementary properties of convergent series.

Theorem B.2.1 (Theorem on series). *Suppose that* $< x_n >, < y_n >$ *are sequences. If the series* $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$ *converge, then*

$$\begin{aligned}
&\sum_{n=1}^{\infty}(x_n + y_n) \text{ converges and } \sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n,\\
&\text{If } a \in \mathbb{R} \sum_{n=1}^{\infty} ax_n = a \sum_{n=1}^{\infty} x_n, \qquad \text{(B.2)}\\
&\text{If } x_n \geq 0 \text{ for all } n, \text{ then } \sum_{n=1}^{\infty} x_n \geq 0.
\end{aligned}$$

There are many criteria, used to determine if a given series converges. The most important is the comparison test.

Theorem B.2.2 (Comparison Test). *Suppose that* $< x_n >, < y_n >$ *are sequences such that, for all* n, $|x_n| \leq y_n$. *If* $\sum_{n=1}^{\infty} y_n$ *converges, then so does* $\sum_{n=1}^{\infty} x_n$. *If* $0 \leq y_n \leq x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *diverges, then so does* $\sum_{n=1}^{\infty} x_n$.

Since this test calls for a comparison, we need to have examples of series that are known to converge or diverge. The simplest case is a geometric series. This is because there is a formula for the partial sums:

$$\sum_{n=0}^{k} a^k = \frac{a^{k+1} - 1}{a - 1}.$$

From this formula we immediately conclude the following:

Theorem B.2.3 (Convergence of geometric series). *A geometric series converges if and only if* $|a| < 1$.

The root and ratio tests are special cases of the comparison test where the series is comparable to a geometric series.

Theorem B.2.4 (Ratio test). *If $< x_n >$ is a sequence with*

$$\limsup_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \alpha,$$

then the series

$$\sum_{n=1}^{\infty} x_n \begin{cases} \textit{converges if} & \alpha < 1 \\ \textit{diverges if} & \alpha > 1. \end{cases}$$

The test gives no information if $\alpha = 1$.

We also have the following:

Theorem B.2.5 (Root test). *If $< x_n >$ is a sequence with*

$$\limsup_{n\to\infty} |x_n|^{\frac{1}{n}} = \alpha,$$

then the series

$$\sum_{n=1}^{\infty} x_n \begin{cases} \textit{converges if} & \alpha < 1 \\ \textit{diverges if} & \alpha > 1. \end{cases}$$

The test gives no information if $\alpha = 1$.

If $\alpha < 1$ in the ratio or root tests, then the series converge absolutely.

Another test is obtained by comparing a series to an integral.

Theorem B.2.6 (Integral test). *If f is a non-negative, monotone decreasing, integrable function defined for $x \geq 1$, then*

$$\sum_{n=1}^{\infty} f(n) \textit{ converges if and only if } \lim_{n\to\infty}\int_1^n f(x)\,dx \textit{ exists.}$$

From this test it follows that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p > 1$. If a series is shown to converge using any of the foregoing tests, then the series converges absolutely. We give a final test that sometimes gives the convergence of a non-absolutely convergent series.

Theorem B.2.7 (Alternating series test). *Suppose that $< x_n >$ is a sequence such that the sign alternates, the $\lim_{n\to\infty} x_n = 0$, and $|x_{n+1}| \leq |x_n|$. Then*

$$\sum_{n=1}^{\infty} x_n$$

converges and

$$\left|\sum_{n=1}^{\infty} x_n - \sum_{n=1}^{N} x_n\right| \leq |x_{N+1}|. \tag{B.3}$$

Note that this test requires that the signs alternate, the sequence of absolute values is monotonely decreasing, and the sequence tends to zero. If any of these conditions are not met, the series may fail to converge.

A useful tool for working with integrals is the integration by parts formula; see Proposition B.6.1. This formula has a discrete analogue, called the *summation by parts formula*, which is important in the study of non-absolutely convergent series.

Proposition B.2.1 (Summation by parts formula). *Let* $< x_n >$ *and* $< y_n >$ *be sequences of numbers. For each* n, *let*

$$Y_n = \sum_{k=1}^{n} y_k;$$

then

$$\sum_{n=1}^{N} x_n y_n = x_N Y_N - \sum_{n=1}^{N-1} (x_{n+1} - x_n) Y_n. \tag{B.4}$$

Using this formula, it is often possible to replace a conditionally convergent sum by an absolutely convergent sum.

***Example* B.2.1.** Let $\alpha = e^{2\pi i x}$, where $x \notin \mathbb{Z}$, so that $\alpha \neq 1$. For any such α, the series

$$\sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n}}$$

converges. To prove this, observe that

$$B_n = \sum_{k=1}^{n} \alpha^k = \frac{\alpha^{n+1} - 1}{\alpha - 1}$$

is a uniformly bounded sequence and

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \leq \frac{1}{n^{\frac{3}{2}}}.$$

The summation by parts formula gives

$$\sum_{n=1}^{N} \frac{\alpha^n}{\sqrt{n}} = \left(\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N+1}} \right) B_N - \sum_{n=1}^{N-1} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) B_n.$$

The boundary term on the right goes to zero as $N \to \infty$, and the sum is absolutely convergent. This shows how the summation by parts formula can be used to convert a conditionally convergent sum into an absolutely convergent sum.

B.3 Limits of Functions and Continuity

The next thing we consider is the behavior of functions defined on intervals in $\mathbb{R}$. Suppose that f is defined for $x \in (a, c) \cup (c, b)$. This is called a *punctured or deleted neighborhood* of c.

***Definition* B.3.1.** We say that the function f has limit L, as x approaches c if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{provided } 0 < |x - c| < \delta;$$

we write

$$\lim_{x \to c} f(x) = L.$$

Note that in this definition nothing is said about the value of f at c. This has no bearing at all on whether or not the limit exists.

***Definition* B.3.2.** If $f(c)$ is defined and we have that

$$\lim_{x \to c} f(x) = f(c),$$

then we say that f is continuous at c. If f is continuous for all $x \in (a, b)$, then we say that f is continuous on (a, b).

In addition to the ordinary limit, we also define one-sided limits. If f is defined in (a, b) and there exists an L such that, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ provided } 0 < x - a < \delta, \text{ then } \lim_{x \to a^+} f(x) = L.$$

If instead

$$|f(x) - L| < \epsilon \text{ provided } 0 < b - x < \delta, \text{ then } \lim_{x \to b^-} f(x) = L.$$

The rules for dealing with limits of functions are very similar to the rules for handling limits of sequences

Theorem B.3.1 (Algebraic rules for limits of functions). *Suppose that f, g are defined in a punctured neighborhood of c and that*

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M.$$

Then

$$\begin{aligned} \lim_{x \to c}(af(x)) &\text{ exists and equals } aL \text{ for all } a \in \mathbb{R}, \\ \lim_{x \to c}(f(x) + g(x)) &\text{ exists and equals } L + M, \\ \lim_{x \to c}(f(x)g(x)) &\text{ exists and equals } LM, \\ \text{provided } M \neq 0, \lim_{x \to c} \frac{f(x)}{g(x)} &\text{ exists and equals } \frac{L}{M}. \end{aligned} \tag{B.5}$$

From this we deduce the following results about continuous functions:

Theorem B.3.2 (Algebraic rules for continuous functions). *If f, g are continuous at c, then so are af, $f+g$, fg. If $g(c) \neq 0$, then f/g is also continuous at c.*

For functions there is one further operation that is very important: composition.

Theorem B.3.3 (Continuity of compositions). *Suppose that $f(x)$, $g(y)$ are two functions such that $f(x)$ is continuous at $x = c$ and $g(y)$ is continuous at $y = f(c)$. Then the composite function, $g \circ f(x)$, is continuous at $x = c$.*

***Definition* B.3.3.** A function defined on an interval $[a, b]$ is said to be *uniformly continuous* if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon, \text{ for all } x, y \in [a, b] \text{ with } |x - y| < \delta.$$

The basic proposition is as follows:

Proposition B.3.1. *A continuous function on a closed, bounded interval is uniformly continuous.*

Using similar arguments, we can also prove the following:

Proposition B.3.2 (Max-min theorem for continuous functions). *If f is continuous on a closed bounded interval, $[a, b]$, then there exists $x_1 \in [a, b]$ and $x_2 \in [a, b]$ that satisfy*

$$f(x_1) = \sup_{x \in [a,b]} f(x), \quad f(x_2) = \inf_{x \in [a,b]} f(x).$$

As a final result on continuous functions, we have the Intermediate value theorem

Theorem B.3.4 (Intermediate value theorem). *Suppose that f is continuous on $[a, b]$ and $f(a) < f(b)$. Then for each $y \in (f(a), f(b))$ there exists $c \in (a, b)$ such that $f(c) = y$.*

B.4 Differentiability

A function defined in a neighborhood of a point c is said to be *differentiable* at c if the function

$$g(x) = \frac{f(x) - f(c)}{x - c},$$

defined in a punctured neighborhood of c, has a limit as $x \to c$. This limit is called the derivative of f at c; we denote it by $f'(c)$. A function that is differentiable at every point of an interval is said to be differentiable in the interval. If the derivative is itself continuous, then the function is said to be *continuously differentiable*. As with continuous functions, we have algebraic rules for differentiation.

Proposition B.4.1 (Rules for differentiation). *Suppose that f, g are differentiable at c; then so are af, $(f+g)$, fg. If $g(c) \neq 0$, then so is f/g. The derivatives are given by*

$$\begin{aligned}(af)'(c) &= a(f'(c)),\\ (f+g)'(c) &= f'(c) + g'(c),\\ (fg)'(c) &= f'(c)g(c) + f(c)g'(c),\\ \left(\frac{f}{g}\right)'(c) &= \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.\end{aligned} \tag{B.6}$$

We can also differentiate a composition.

Proposition B.4.2 (The chain rule). *If $f(x)$ is differentiable at $x = c$ and $g(y)$ is differentiable at $y = f(c)$, then $g \circ f(x)$ is differentiable at $x = c$; the derivative is*

$$g \circ f'(c) = g'(f(c))f'(c).$$

It is often useful to be able to compare the sizes of two functions f, g near a point c without being too specific. The *big O* and *little o* notations are often used for this purpose.

***Definition* B.4.1.** The notation

$$f(x) = O(g(x)) \qquad \text{near to } c$$

means that there exists an M and an $\epsilon > 0$ such that

$$|f(x)| < Mg(x) \qquad \text{provided } |x - c| < \epsilon,$$

whereas

$$f(x) = o(g(x)) \qquad \text{near to } c$$

means that

$$\lim_{x \to c} \frac{|f(x)|}{g(x)} = 0.$$

For example, a function f is differentiable at c if and only if there exists a number L for which

$$f(x) = f(c) + L(x - c) + o(|x - c|).$$

Of course, $L = f'(c)$. This implies that a function that is differentiable at a point is also continuous at that point. The converse statement is false: A function may be continuous at a point without being differentiable; for example, $f(x) = |x|$ is continuous at 0 but not differentiable.

B.5 Higher-Order Derivatives and Taylor's Theorem

If the first derivative of function, f', is also differentiable, then we say that f is twice differentiable. The second derivative is denoted by f''. Inductively, if the kth derivative is differentiable, then we say that f is $(k+1)$ times differentiable. The kth derivative of f is denoted by $f^{[k]}$. For a function that has n derivatives, there is a polynomial that agrees with f to order $n-1$ at a point.

Theorem B.5.1 (Taylor's Theorem). *Suppose that f has n derivatives in an interval $[a,b]$. If $x, c \in (a,b)$, then*

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{[j]}(c)(x-c)^j}{j!} + R_n(x), \tag{B.7}$$

where

$$R_n(x) = O(|x-c|^n).$$

Formula (B.7) is called *Taylor's formula with remainder term*. There are many different formulæ for the *remainder term R_n*. One from which all the others can be derived is given by

$$R_n(x) = \frac{1}{(n-1)!} \int_c^x f^{[n]}(t)(x-t)^{n-1} dt. \tag{B.8}$$

The $n = 1$ case of Taylor's theorem is the mean value theorem.

Theorem B.5.2 (Mean value theorem). *Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

B.6 Integration

The inverse operation to differentiation is integration. Suppose that f is a bounded function defined on a finite interval $[a,b]$. An increasing sequence $P = \{a = x_0 < x_1 < \cdots < x_N = b\}$ defines a *partition* of the interval. The mesh size of the partition is defined to be

$$|P| = \max\{|x_i - x_{i-1}| \ : \ i = 1, \ldots, N\}.$$

To each partition we associate two approximations of the area under the graph of f, by the rules

$$\begin{aligned} U(f,P) &= \sum_{j=1}^{N} \sup_{x \in [x_{j-1}, x_j]} f(x)(x_j - x_{j-1}), \\ L(f,P) &= \sum_{j=1}^{N} \inf_{x \in [x_{j-1}, x_j]} f(x)(x_j - x_{j-1}). \end{aligned} \tag{B.9}$$

These are called the *upper and lower Riemann sums*. Observe that for any partition P we have the estimate

$$U(f, P) \geq L(f, P). \tag{B.10}$$

If P and P' are partitions with the property that every point in P is also a point in P', then we say that P' is a *refinement* of P and write $P < P'$. If P_1 and P_2 are two partitions, then, by using the union of the points in the two underlying sets, we can define a new partition P_3 with the property that

$$P_1 < P_3 \text{ and } P_2 < P_3.$$

A partition with this property is called a *common refinement* of P_1 and P_2. From the definitions it is clear that if $P < P'$, then

$$U(f, P) \geq U(f, P') \text{ and } L(f, P) \leq L(f, P'). \tag{B.11}$$

***Definition* B.6.1.** A bounded function f defined on an interval $[a, b]$ is *Riemann integrable* if

$$\inf_P U(f, P) = \sup_P L(f, P).$$

In this case we denote the common value, called the *Riemann integral*, by

$$\int_a^b f(x)\,dx.$$

Most "nice" functions are Riemann integrable. For example we have the following basic result.

Theorem B.6.1. *Suppose that f is a piecewise continuous function defined on $[a, b]$. Then f is Riemann integrable and*

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{j=1}^{N} f\Big(a + \frac{j}{N}(b-a)\Big)\frac{b-a}{N}.$$

The proof of this theorem is not difficult and relies primarily on the uniform continuity of a continuous function on a closed, bounded interval and (B.11). The sums appearing in this theorem are called *right Riemann sums*, because the function is evaluated at the right endpoint of each interval. The *left Riemann sums* are obtained by evaluating at the left endpoints. The formula for the integral holds for any Riemann integrable function but is more difficult to prove in this generality. The integral is a linear map from integrable functions to the real numbers.

Theorem B.6.2. *Suppose that f and g are Riemann integrable functions. Then $f+g$ and fg are integrable as well. If $c \in \mathbb{R}$, then*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \text{ and } \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Theorem B.6.3. *Suppose that f is Riemann integrable on $[a,b]$ and that $c \in [a,b]$. Then f is Riemann integrable on $[a,c]$ and $[c,b]$; moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{B.12}$$

There is also a mean value theorem for the integral, similar to Theorem B.5.2.

Theorem B.6.4. *Suppose that f is a continuous function and w is a nonnegative integrable function. There exists a point $c \in (a,b)$ so that*

$$\int_a^b f(x)w(x)\,dx = f(c)\int_a^b w(x)\,dx.$$

The mean theorem provides a different formula for the remainder term in Taylor's theorem.

Corollary B.6.1. *Suppose that f has n-derivatives on an interval $[a,b]$ and x, c are points in (a,b). Then there exists a number d, between x and c, so that*

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{[j]}(c)(x-c)^j}{j!} + \frac{f^{[n]}(d)(x-c)^n}{n!}. \tag{B.13}$$

Most elementary methods for calculating integrals come from the fundamental theorem of calculus. To state this result we need to think of the integral in a different way. As described previously, the integral associates a number to a function defined on a fixed interval. Suppose instead that f is defined and Riemann integrable on $[a,b]$. Theorem B.6.3 states that, for each $x \in [a,b]$, f is Riemann integrable on $[a,x]$. The new idea is to use the integral to define a new *function* on $[a,b]$ by setting

$$F(x) = \int_a^x f(y)\,dy.$$

This function is called the *indefinite integral* or anti-derivative of f. In this context we often refer to $\int_a^b f(x)\,dx$ as the *definite integral* of f.

Theorem B.6.5 (The Fundamental theorem of calculus). *If f is a continuous function on $[a, b]$ then F is differentiable and $F' = f$. If f is differentiable and f' is Riemann integrable then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

There are two further basics tools needed to compute and manipulate integrals. The first is called *integration by parts*, it is a consequence of the product rule for derivatives; see Proposition B.4.1.

Proposition B.6.1 (Integration by parts). *If $f, g \in \mathscr{C}^1([a, b])$ then*

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

The other formula follows from the chain rule, Proposition B.4.2.

Proposition B.6.2 (Change of variable). *Let g be a monotone increasing, differentiable function defined $[a, b]$ with $g(a) = c$, $g(b) = d$ and let f be a Riemann integrable function on $[c, d]$. The following formula holds:*

$$\int_{g(a)}^{g(b)} f(y)\,dy = \int_a^b f(g(x))g'(x)\,dx.$$

B.7 Improper Integrals

In the previous section we defined the Riemann integral for *bounded* functions on *bounded* intervals. In applications both of these restrictions need to be removed. This leads to various notions of *improper integrals*. The simplest situation is that of a function f defined on $[0, \infty)$ and integrable on $[0, R]$ for every $R > 0$. We say that the improper integral,

$$\int_0^\infty f(x)\,dx$$

exists if the limit,

$$\lim_{R\to\infty} \int_0^R f(x)\,dx, \tag{B.14}$$

exists. In this case the improper integral is given by the limiting value. By analogy with the theory of infinite series, there are two distinct situations in which the improper integral exists. If the improper integral of $|f|$ exists, then we say that f is *absolutely integrable* on $[0, \infty)$.

***Example* B.7.1.** The function $(1+x^2)^{-1}$ is absolutely integrable on $[0, \infty)$. Indeed we see that if $R < R'$, then

$$\begin{aligned} 0 \leq \int_0^{R'} \frac{dx}{1+x^2} - \int_0^{R} \frac{dx}{1+x^2} &= \int_R^{R'} \frac{dx}{1+x^2} \\ &\leq \int_R^{R'} \frac{dx}{x^2} \\ &\leq \frac{1}{R}. \end{aligned} \tag{B.15}$$

This shows that

$$\lim_{R\to\infty} \int_0^R \frac{dx}{1+x^2}$$

exists.

***Example* B.7.2.** The function $\frac{\sin x}{x}$ is a bounded continuous function; it is integrable on $[0, \infty)$ but not absolutely integrable. The integral of $\frac{\sin x}{x}$ over any finite interval is finite. Using integration by parts, we find that

$$\int_1^R \frac{\sin x dx}{x} = \frac{\cos x}{x}\Big|_1^R - \int_1^R \frac{\cos x dx}{x^2}.$$

Using this formula and the previous example, it is not difficult to show that

$$\lim_{R\to\infty} \int_0^R \frac{\sin x dx}{x}$$

exists. On the other hand because

$$\int_1^R \frac{dx}{x} = \log R,$$

it is not difficult to show that

$$\int_0^R \frac{|\sin x| dx}{x}$$

grows like $\log R$ and therefore diverges as R tend to infinity.

There are similar definitions for the improper integrals

$$\int_{-\infty}^{0} f(x)\,dx \text{ and } \int_{-\infty}^{\infty} f(x)\,dx.$$

The only small subtlety is that we say that the improper integral exists in the second case only when both the improper integrals,

$$\int_{-\infty}^{0} f(x)\,dx \text{ and } \int_{0}^{\infty} f(x)\,dx,$$

exist separately. Similar definitions apply to functions defined on bounded intervals (a, b) that are integrable on any subinterval $[c, d]$. We say that the improper integral

$$\int_{a}^{b} f(x)\,dx$$

exists if the limits

$$\lim_{c\to a^+} \int_{c}^{e} f(x)\,dx \text{ and } \lim_{c\to b^-} \int_{e}^{c} f(x)\,dx$$

both exist. Here e is any point in (a, b); the existence or nonexistence of these limits is clearly independent of which (fixed) point we use. Because improper integrals are defined by limits of proper integrals, they have the same linearity properties as integrals. For example,

Proposition B.7.1. *Suppose that f and g are improperly integrable on $[0, \infty)$. Then $f + g$ is as well and*

$$\int_{0}^{\infty} (f(x) + g(x))\,dx = \int_{0}^{\infty} f(x)\,dx + \int_{0}^{\infty} g(x)\,dx,$$

for $a \in \mathbb{R}$, af is improperly integrable and

$$\int_{0}^{\infty} af(x)\,dx = a\int_{0}^{\infty} f(x)\,dx.$$

The final case that requires consideration is that of a function f defined on a punctured interval $[a, b) \cup (b, c]$ and integrable on subintervals of the form $[a, e]$ and $[f, c]$, where $a \le e < b$ and $b < f \le c$. If both limits

$$\lim_{e\to b^-} \int_{a}^{e} f(x)\,dx \text{ and } \lim_{f\to b^+} \int_{f}^{c} f(x)\,dx$$

exist, then we say that f is improperly integrable on $[a, b]$. For example, the function $f(x) = x^{-\frac{1}{3}}$ is improperly integrable on $[-1, 1]$. On the other hand, the function $f(x) = x^{-1}$ is not improperly integrable on $[-1, 1]$ because

$$\lim_{e \to 0^+} \int_e^1 \frac{dx}{x} = \infty \text{ and } \lim_{f \to 0^-} \int_{-1}^f \frac{dx}{x} = -\infty.$$

There is a further extension of the notion of integrability that allows us to assign a meaning to

$$\int_{-1}^{1} \frac{dx}{x}.$$

This is called the *principal value integral* or *Cauchy principal value integral.* The observation is that for any $\epsilon > 0$

$$\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} = 0,$$

so the limit of this sum of integrals exists as ϵ goes to zero.

***Definition* B.7.1.** Suppose that f is defined on the punctured interval $[a, b) \cup (b, c]$ and is integrable on any subinterval $[a, e]$, $a \leq e < b$ or $[f, c]$, $b < f \leq c$. If the limit

$$\lim_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)\, dx + \int_{b+\epsilon}^{c} f(x)\, dx$$

exists, then we say that f has a *principal value integral* on $[a, c]$. We denote the limit by

$$\text{P.V.} \int_a^c f(x)\, dx.$$

For a function that is not (improperly) integrable on $[a, b]$, the principal value integral exists because of cancellation between the divergences of the two parts of the integral. The approach to the singular point is symmetric; both the existence of the limit and its value depend crucially on this fact.

***Example* B.7.3.** We observed that the function x^{-1} has a principal value integral on $[-1, 1]$ and its value is zero. To see the importance of symmetry in the definition of the principal value integral, observe that

$$\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{2\epsilon}^{1} \frac{dx}{x} = -\log 2$$

and

$$\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon^2}^{1} \frac{dx}{x} = -\log \epsilon.$$

In the first case we get a different limit and in the second case the limit does not exist.

The material in this chapter is usually covered in an undergraduate course in mathematical analysis. The proofs of these results and additional material can be found in [27], [111], and [121].

B.8 Fubini's Theorem and Differentiation of Integrals*

This section contains two results of a more advanced character than those considered in the previous sections of this appendix. This material is included because these results are used many times in the main body of the text.

There is a theory of integration for functions of several variables closely patterned on the one -variable case. A rectangle in $\mathbb{R}^n$ is a product of bounded intervals

$$R = [a_1, b_1) \times \cdots \times [a_n, b_n).$$

The n-dimensional volume of R is defined to be

$$|R| = \prod_{j=1}^{n} (b_j - a_j).$$

Suppose that f is a bounded function with bounded support in $\mathbb{R}^n$. A partition of the support of f is a collection of disjoint rectangles $\{R_1, \ldots, R_N\}$ such that

$$\operatorname{supp} f \subset \bigcup_{j=1}^{N} R_j.$$

To each partition P of supp f we associate an upper and lower Riemann sum:

$$U(f, P) = \sum_{j=1}^{N} \sup_{\boldsymbol{x} \in R_j} f(\boldsymbol{x})|R_j|, \quad L(f, P) = \sum_{j=1}^{N} \inf_{\boldsymbol{x} \in R_j} f(\boldsymbol{x})|R_j|.$$

As before, f is integrable if

$$\inf_P U(f, P) = \sup_P L(f, P).$$

In this case the integral of f over $\mathbb{R}^n$ is denoted by

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Let B_r denote the ball centered at zero of radius r. If $\chi_{B_r}|f|$ is integrable for every r and

$$\lim_{r\to\infty}\int_{\mathbb{R}^n}\chi_{B_r}|f|(\boldsymbol{x})\,d\boldsymbol{x}$$

exists, then we say that f is absolutely integrable on $\mathbb{R}^n$. It is not difficult to extend the definition of absolute integrability to unbounded functions. Let f be a function defined on $\mathbb{R}^n$ and set

$$E_R = f^{-1}([-R, R]).$$

Suppose that for every positive number R the function $\chi_{E_R} f$ is absolutely integrable on $\mathbb{R}^n$. If the limit

$$\lim_{R\to\infty}\int_{\mathbb{R}^n}\chi_{E_R}|f|(\boldsymbol{x})\,d\boldsymbol{x}$$

exists, then we say that f is absolutely integrable and let

$$\int_{\mathbb{R}^n} f(\boldsymbol{x})\,d\boldsymbol{x} = \lim_{R\to\infty}\int_{\mathbb{R}^n}\chi_{E_R} f(\boldsymbol{x})\,d\boldsymbol{x}.$$

Suppose that $n = k + l$ for two positive integers k and l. Then $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$. Let $\boldsymbol{w}$ be coordinates for $\mathbb{R}^k$ and $\boldsymbol{y}$ coordinates for $\mathbb{R}^l$. Assume that for each $\boldsymbol{w}$ in $\mathbb{R}^k$ the function $f(\boldsymbol{w}, \cdot)$ on $\mathbb{R}^l$ is absolutely integrable and the function

$$g(\boldsymbol{w}) = \int_{\mathbb{R}^l} f(\boldsymbol{w}, \boldsymbol{y})\,d\boldsymbol{y}$$

is an integrable function on $\mathbb{R}^k$. The integral of g over $\mathbb{R}^k$ is an *iterated integral* of f; it usually expressed as

$$\int_{\mathbb{R}^n} g(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\mathbb{R}^k}\int_{\mathbb{R}^l} f(\boldsymbol{w}, \boldsymbol{y})\,d\boldsymbol{w}\,d\boldsymbol{y}.$$

It is reasonable to enquire how is the integral of g over $\mathbb{R}^k$ is related to the integral of f over $\mathbb{R}^n$. Fubini's theorem provides a comprehensive answer to this question.

Theorem B.8.1 (Fubini's theorem). *Let f be a function defined on $\mathbb{R}^n$ and let $n = k + l$ for positive integers k and l. If either of the iterated integrals*

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^l} |f(\boldsymbol{w}, \boldsymbol{y})|\,d\boldsymbol{w}\,d\boldsymbol{y} \text{ or } \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} |f(\boldsymbol{w}, \boldsymbol{y})|\,d\boldsymbol{y}\,d\boldsymbol{w}$$

is finite, then the other is as well. In this case f is integrable over $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^l} f(\boldsymbol{w}, \boldsymbol{y})\,d\boldsymbol{w}\,d\boldsymbol{y} = \int_{\mathbb{R}^n} f(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} f(\boldsymbol{w}, \boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{w} \tag{B.16}$$

Informally, the order of the integrations can be interchanged. Note that we assume that f is *absolutely* integrable in order to conclude that the order of integrations of f can be interchanged. There are examples of functions defined on $\mathbb{R}^2$ so that both iterated integrals,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy \text{ and } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dy\, dx$$

exist but are unequal, and f is not integrable on $\mathbb{R}^2$. A proof of Fubini's theorem can be found in [121] or [43].

The second problem we need to consider is that of differentiation under the integral sign. For a positive number ϵ, let f be a function defined on $\mathbb{R}^n \times (a - \epsilon, a + \epsilon)$. Suppose that for each y in $(a - \epsilon, a + \epsilon)$, the function $f(\cdot, y)$ is absolutely integrable on $\mathbb{R}^n$, and for each $\boldsymbol{x}$ in $\mathbb{R}^n$, the function $f(\boldsymbol{x}, \cdot)$ is differentiable at a. Is the function defined by the integral

$$g(y) = \int_{\mathbb{R}^n} f(\boldsymbol{x}, y)\, d\boldsymbol{x}$$

differentiable at a? In order for this to be true, we need to assume that the difference quotients

$$\frac{f(\boldsymbol{x}, a + h) - f(\boldsymbol{x}, a)}{h}$$

satisfy some sort of uniform bound. The following theorem is sufficient for our applications.

Theorem B.8.2. *With f as before, if there exists an absolutely integrable function F so that for every h with $|h| < \epsilon$, we have the estimate*

$$\left| \frac{f(\boldsymbol{x}, a + h) - f(\boldsymbol{x}, a)}{h} \right| \leq F(\boldsymbol{x}),$$

then

$$g(y) = \int_{\mathbb{R}^n} f(\boldsymbol{x}, y)\, d\boldsymbol{x}$$

is differentiable at a and

$$g' = \int_{\mathbb{R}^n} \partial_y f(\boldsymbol{x}, y)\, d\boldsymbol{x}.$$

This theorem is a consequence of the Lebesgue dominated convergence theorem.