

AMCS 608 Problem Set 1  
due September 22, 2009  
Dr. Epstein

**Reading:** References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *Elementary Classical Analysis*, by Jerrold Marsden, *Calculus on Manifolds* by Michael Spivak, *Linear Algebra*, by Peter Lax.

**Standard problems:** The following problems should be done, but do not have to be handed in.

1. Let  $V$  be a real vector space, prove that every basis for  $V$  has the same number of elements (that is  $\dim V$  is well defined).
2. Let  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear transformation; show that
  - (a)  $\text{Im } A$  and  $\ker A$  are subspaces of  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively.
  - (b) Show that  $\mathbb{R}^m / \ker A$  is a vector space.
  - (c) Show that  $\mathbb{R}^m / \ker A$  is isomorphic to  $\text{Im } A$ .
3. Let  $\langle x_n \rangle$  be a bounded sequence of real numbers. Show that there are increasing sequences of positive integers  $\langle p_j \rangle$ ,  $\langle q_j \rangle$  so that

$$\limsup_{n \rightarrow \infty} x_n = \lim_{j \rightarrow \infty} x_{p_j} \text{ and } \liminf_{n \rightarrow \infty} x_n = \lim_{j \rightarrow \infty} x_{q_j}. \quad (1)$$

4. Suppose that  $W, V$  are subspaces of  $\mathbb{R}^n$ , and  $\langle \cdot, \cdot \rangle$  is an inner product on  $\mathbb{R}^n$ . Define

$$V^\perp = \{x \in \mathbb{R}^n : \langle x, v \rangle = 0 \text{ for all } v \in V\}. \quad (2)$$

- (a) Show that  $V^\perp$  is a subspace.
- (b) Show that  $\mathbb{R}^n \simeq V \oplus V^\perp$ , hence  $\dim V^\perp = n - \dim V$ .
- (c) Show that if  $W \subset V^\perp$ , then  $\dim W \leq \dim V^\perp$ .

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Let  $V$  be a finite dimensional  $\mathbb{R}$ -vector space and  $V'$  the space of linear functions on  $V$ . Give an algebraic proof that  $\dim V = \dim V'$ .

2. Let

$$\langle x, y \rangle = \sum_{j=1}^n x_j y_j \quad (3)$$

be the standard inner product on  $\mathbb{R}^n$ . If  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a linear transform, then the transpose of  $A$  with respect to this inner product,  $A^t$ , is defined by the condition

$$\langle Ax, y \rangle = \langle x, A^t y \rangle. \quad (4)$$

Let  $B = (b_{ij})$  be a positive definite symmetric matrix, that is  $b_{ij} = b_{ji}$ , and

$$\sum_{i,j=1}^n b_{ij} x_i x_j \geq 0, \quad (5)$$

with equality if and only if  $x = 0$ .

(a) Show that

$$(x, y) = \sum_{i,j=1}^n b_{ij} x_i y_j \quad (6)$$

defines another inner product on  $\mathbb{R}^n$ .

(b) We can define a different notion of transpose,  $A^\dagger$  by the requirement that

$$(Ax, y) = (x, A^\dagger y) \quad (7)$$

How is  $A^\dagger$  related to  $A^t$ ?

(c) Show directly that  $Ax = y$  is solvable if and only if  $y$  is  $(\cdot, \cdot)$ -orthogonal to the  $\ker A^\dagger$ .

3. Let  $\mathcal{P}_2$  be the space of polynomials of degree at most 2. For  $p, q \in \mathcal{P}_2$ , define

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx. \quad (8)$$

(a) Prove that this defines an inner product on  $\mathcal{P}_2$ .

(b) For  $k = 0, 1, 2$ , show that  $\ell_k(p) = \partial_x^k p(0)$  define elements of  $\mathcal{P}'_2$ .

(c) For  $k = 0, 1, 2$  find elements  $q_k \in \mathcal{P}_2$  so that

$$\ell_k(p) = \langle p, q_k \rangle. \quad (9)$$

(d) If  $p$  is a polynomial of degree 3 is it true that

$$\ell_k(p) = \langle p, q_k \rangle? \quad (10)$$

4. For each  $p \in [0, \infty)$  we define a function on  $\mathbb{R}^n$  by

$$N_p(x) = \left[ \sum_{j=1}^n |x_j|^p \right]^{\frac{1}{p}}. \quad (11)$$

For  $p = \infty$  we let:

$$N_\infty(x) = \max\{|x_1|, \dots, |x_n|\}. \quad (12)$$

- (a) Show that  $\lim_{p \rightarrow \infty} N_p(x) = N_\infty(x)$ .
- (b) For  $p = 1, 2, \infty$  and  $n = 2$  draw the unit balls,  $\{x : N_p(x) \leq 1\}$  on a single graph.
- (c) Show, for  $n = 2$ , that  $N_{\frac{1}{2}}(x)$  does **not** define a norm on  $\mathbb{R}^2$ . Draw the unit ball.
- (d) For each  $n$  find constants  $A_n, B_n$  so that, for  $x \in \mathbb{R}^n$  we have

$$A_n N_\infty(x) \leq N_1(x) \leq B_n N_\infty(x). \quad (13)$$

5. A set  $A \subset \mathbb{R}$  is *not* connected if there exist disjoint open sets  $U, V$  such that
- (a)  $U \cap A \neq \emptyset$  and  $V \cap A \neq \emptyset$
  - (b)  $A \subset U \cup V$ .

What are the connected subsets of  $\mathbb{R}$ ? You must prove your answer.

6. Suppose that  $f$  is a differentiable function on  $(0, \infty)$ . Show that

$$\sum_{n=1}^N f(n) = \int_1^{N+1} f(x) dx - \sum_{n=1}^N \int_n^{n+1} f'(x)(n+1-x) dx. \quad (14)$$

Use this formula to prove that there is a non-zero constant  $\gamma$  so that

$$\sum_{n=1}^N \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right). \quad (15)$$

Is  $\gamma$  positive or negative?